

LINEAR REPRESENTATIONS OF REGULAR RINGS AND COMPLEMENTED MODULAR LATTICES WITH INVOLUTION

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ABSTRACT. Faithful representations of regular \ast -rings and modular complemented lattices with involution within orthosymmetric sesquilinear spaces are studied within the framework of Universal Algebra. In particular, the correspondence between classes of spaces and classes of representable structures is analyzed; for a class \mathcal{S} of spaces which is closed under ultraproducts and non-degenerate finite-dimensional subspaces, the class of representable structures is shown to be closed under complemented [regular] subalgebras, homomorphic images, and ultraproducts. Moreover, this class is generated by its members which are isomorphic to subspace lattices with involution [endomorphism \ast -rings, respectively] of finite-dimensional spaces from \mathcal{S} . Under natural restrictions, this result is refined to a 1-1-correspondence between the two types of classes.

1. INTRODUCTION

For \ast -rings, there is a natural and well established concept of representation in a vector space V_F endowed with an orthosymmetric sesquilinear form: a homomorphism ε into the endomorphism ring of V_F such that $\varepsilon(r^\ast)$ is the adjoint of $\varepsilon(r)$. Famous examples of [faithful] representations are due to Gel'fand-Naimark-Segal (C^\ast -algebras in Hilbert space) and Kaplansky (primitive \ast -rings with a minimal right ideal), cf. [2, Theorem 4.6.6].

[Faithful] representability of \ast -regular rings within anisotropic inner product spaces has been studied by Micol [44] and used to derive results in the universal algebraic theory of these structures. For the \ast -regular rings of classical quotients of finite Rickart C^\ast -algebras (cf. Ara and Menal [1]), existence of representations has been established in [30]. For complemented modular lattices with involution $a \mapsto a'$ (CMILs for short), an analogue of the concept of representation is a lattice homomorphism ε , preserving the bounds 0 and 1, into the lattice of all subspaces such that $\varepsilon(a')$ is the orthogonal subspace to $\varepsilon(a)$ (cf. Niemann [46]). The latter has been considered in the context of synthetic orthogeometries in [22], continuing earlier work on anisotropic geometries and modular ortholattices [25, 26, 27]. Primary examples are atomic CMILs associated with irreducible desarguean orthogeometries and those CMILs which arise from lattices of principal right ideals of representable regular \ast -rings.

The [proofs of the] main results of these studies relate closure properties of a class \mathcal{S} of spaces with closure properties of the class \mathcal{R} of algebraic structures [faithfully] representable within spaces from \mathcal{S} . In particular, for a class \mathcal{S} closed under ultraproducts and non-degenerate finite-dimensional subspaces, one has \mathcal{R} closed under

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ultraproducts, homomorphic images, and regular [complemented, respectively] subalgebras. Moreover, with an approach due to Tyukavkin [50], it has been shown that \mathcal{R} is generated, with respect to these operators, by the endomorphism \ast -rings [by the subspace lattices with involution $U \mapsto U^\perp$, respectively] of finite-dimensional spaces from \mathcal{S} (cf. Theorem 11.3). Conversely, any class \mathcal{R} of structures generated in this way has its members representable within \mathcal{S} .

The first purpose of the present paper is to extend these results to regular \ast -rings on one hand, to representations within orthosymmetric sesquilinear spaces on the other – thus allowing regular rings with an involution which may have $r^\ast r = 0$ for some $r \neq 0$, that is regular \ast -rings which are not \ast -regular; for example, \ast -rings associated with finite dimensional spaces having some isotropic points. The second one is to give a more transparent presentation by dealing with types of classes naturally associated with representations in linear spaces. We call a class of structures \mathcal{R} as above an \exists -*semivariety* of regular \ast -rings [CMILs] and we call \mathcal{S} a *semivariety* of spaces. The quantifier ‘ \exists ’ refers to the required existence of quasi-inverses [complements, respectively]. In this setting, the above-mentioned relationship between classes of spaces \mathcal{S} and classes of representable structures \mathcal{R} can be refined to a 1-1-correspondence (cf. Theorem 11.6). Also, we observe that \mathcal{R} remains unchanged if \mathcal{S} is enlarged by forming two-sorted substructures, corresponding to the subgeometries in the sense of [22], (cf. Theorem 11.3). We also provide a useful condition on \mathcal{S} which implies that \mathcal{R} is an \exists -*variety*, i.e. that \mathcal{R} is also closed under direct products (see Proposition 12.1). For a reference in later applications, e.g. to decidability results refining those of [23], we consider \ast -rings which are also algebras over a fixed commutative \ast -ring.

In the context of synthetic orthogeometries, the class \mathcal{R} of representable structures is an \exists -variety if \mathcal{S} is also closed under orthogonal disjoint unions. No such natural construction is available for sesquilinear spaces. The alternative, chosen by Micol [44], was to generalize the concept of faithful representation to residually faithful representation; thus, associating with any semivariety of spaces an \exists -variety of generalized representables. We derive these results in our more general setting (cf. Proposition 12.3).

We first present background on sesquilinear spaces (Section 2), rings (Sections 3–4), and lattices (Sections 5,7). Synthetic orthogeometries are included (section 6) for use of the results in [22]. A key to results on representations is to view them as multi-sorted structures (Section 8). The Universal Algebra point of view and the class operators are introduced in Section 9. The basic reduction to finite dimensions is in Section 10, applications to correspondences between classes in Section 11. Section 12 relates these to Micol’s [44] the more general concept of representation. In Sections 8–12 results on rings and on lattices are presented in parallel. Proofs of the former do not depend on the latter. Though, the other way round, we have to use basic results on lattices of principal right ideals of regular rings.

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2. ε -HERMITIAN SPACES

We first define the linear structures providing representations both for lattices and rings with involution. For any division ring F , endowed with an anti-automorphism ν (we write $\nu(\lambda) = \lambda^\nu$) we consider *sesquilinear spaces* which are [right] vector spaces

V_F endowed with a *scalar product* or a *sesquilinear form* $\langle \mid \rangle: V \times V \rightarrow F$; that is, for all $u, v, w \in V$ and all $\lambda, \mu \in F$, one has

$$\langle u \mid v + w \rangle = \langle u \mid v \rangle + \langle u \mid w \rangle, \quad \langle u\lambda \mid v\mu \rangle = \lambda^\nu \langle u \mid v \rangle \mu,$$

Our basic reference is [20, Chapter I] (though, we use “sesquilinear space” in a more general meaning). Observe that, from a right vector space V_F one obtains a left vector space ${}_F V$ putting $\lambda v = v\lambda^{\nu^{-1}}$. By this, a sesquilinear form on V_F as defined above turns out a sesquilinear form in the sense of [20] on ${}_F V$ – both with respect to ν . This gives access to results of [20] in the left vector space setting. Introductions to orthogonal geometry in infinite dimension are also given in [39], [12, Chapter 14], cf. [33, Chapter IV], [31, §1.21], [2, §4.6].

Given a second sesquilinear space $V'_{F'}$ with ν' and $\langle \mid \rangle'$, we have the following concepts relating it with the first: An *isomorphism* between the sesquilinear spaces is a bijection $\omega: V \rightarrow V'$ which is an α -semilinear map $V_F \rightarrow V'_{F'}$ for some isomorphism $\alpha: F \rightarrow F'$ such that $\alpha \circ \nu = \nu' \circ \alpha$ and $\langle \omega(v) \mid \omega(w) \rangle' = \alpha(\langle v \mid w \rangle)$ for all $v, w \in V$. The second space arises from the first by *scaling* with $\mu \in F$ if $F' = F$ and $V_F = V'_{F'}$ as vector spaces, and if $\mu \neq 0$, $r^{\nu'} = \mu r^\nu \mu^{-1}$, and $\langle u \mid v \rangle' = \mu \langle u \mid v \rangle$. Finally, V_F and $V'_{F'}$ are *similar*, if one arises from the other by any composition of isomorphisms and scalings. It is easy to see that any similitude can be expressed as an isomorphism followed by a scaling.

Any vector space V over a division ring F with anti-automorphism ν can be turned into a non-degenerate sesquilinear space: given a basis $v_i (i \in I)$ and $0 \neq \delta_i \in F$ define $\langle \sum_{i \in J} v_i \lambda_i \mid \sum_{i \in J} v_i \mu_i \rangle = \sum_{i \in J} \lambda_i^\nu \delta_i \mu_i$ for finite $J \subseteq I$ and $\lambda_i, \mu_i \in F$. Though, these examples are far from being exhaustive.

Since we consider only one anti-automorphism ν on F and only one scalar product on V_F at a time, we use F to include ν (and write $\lambda^\nu = \lambda^*$) and V_F to denote the space endowed with the scalar product.

A sesquilinear space $V_F \neq 0$ is *non-degenerate* if $\langle u \mid v \rangle = 0$ for all $v \in V$ implies $u = 0$. For $\varepsilon \in F$, V_F is ε -*hermitian* if $\langle v \mid u \rangle = \varepsilon \cdot \langle u \mid v \rangle^*$ for all $u, v \in V$; V_F is *hermitian* if it is 1-hermitian; V_F is *skew symmetric* if it is (-1) -hermitian and $\lambda^* = \lambda$ for all $\lambda \in F$; V_F is *alternate*, if $\langle v \mid v \rangle = 0$ for all $v \in V$ (observe that [2, §4.6] requires characteristic $\neq 2$). V_F is *anisotropic* if $\langle v \mid v \rangle \neq 0$ for all $v \in V$, $v \neq 0$.

For endomorphisms φ, ψ of the vector space V_F we say that ψ is an *adjoint* of φ if $\langle \varphi(u) \mid v \rangle = \langle u \mid \psi(v) \rangle$ for all $u, v \in V$. If V_F is non-degenerate, then any endomorphism φ has at most one adjoint ψ ; if such ψ exists, we write $\psi = \varphi^*$. If φ^* and χ^* exist, then $(\chi \circ \varphi)^* = \varphi^* \circ \chi^*$. The space V_F is *orthosymmetric*, or *reflexive*, if \perp is a symmetric relation.

Proposition 2.1. *The relations of orthogonality and adjointness are left unchanged under scaling; in particular, orthosymmetry is preserved under scaling. Consider a non-degenerate sesquilinear space V_F . The following are equivalent if $\dim V_F > 1$:*

- (i) *the sesquilinear space V_F is orthosymmetric;*
- (ii) *the sesquilinear space V_F is ε -hermitian for some (unique) $\varepsilon \in F \setminus \{0\}$;*
- (iii) *up to scaling, V_F is either hermitian or skew-symmetric;*
- (iv) *the adjointness relation is symmetric on $\text{End}(V_F)$;*
- (v) *if φ^* exists then $\varphi^{**} = \varphi$.*

Furthermore, if V_F is ε -hermitian and non-degenerate, then $\lambda \mapsto \lambda^*$ is an involution on F , that is $(\lambda^*)^* = \lambda$ for all $\lambda \in F$. If V_F is alternate and non-degenerate then

it is skew symmetric and F is commutative; moreover, any $V_{F'}$ similar to V_F is alternate, too.

Proof. The first statement is obviously true. Now, assume V_F non-degenerate and $\dim V_F > 1$. The following references are to [20, Chapter I]. (i) implies (ii) by Theorem 1 of §1.3. (ii) implies (iii) by (15) of §1.5. (iii) implies (i), obviously, proving pairwise equivalence of (i), (ii), and (iii).

Assuming (iii), symmetry of adjointness follows, easily. That, in turn, implies $\varphi = \varphi^{**}$ for every $\varphi \in \text{End}^*(V_F)$. Thus, we have (iii) \Rightarrow (iv) and (iv) \Rightarrow (v).

Assuming (v), let $\lambda \mapsto \lambda^+$ denote the inverse of $\lambda \mapsto \lambda^*$. Given $u \in V$ such that $\mu = \langle u | u \rangle \neq 0$, consider two linear maps:

$$\varphi_u(v) = u(\langle v | u \rangle \mu^{-1})^+ \text{ and } \psi_u(w) = u\mu^{-1}\langle u | w \rangle, \quad v, w \in V.$$

Observe that $\psi_u = \varphi_u^*$, whence by our hypothesis, $\varphi_u = \psi_u^*$. Moreover, φ_u and ψ_u are the projections onto uF associated with the decompositions $V = uF \oplus \{v \in V \mid \langle v | u \rangle = 0\}$ and $V = uF \oplus \{w \in V \mid \langle u | w \rangle = 0\}$, respectively. Now, $\varphi_u = \psi_u \circ \varphi_u$ and we get $\psi_u = \varphi_u^* = \varphi_u^* \circ \psi_u^* = \varphi_u$, an orthogonal projection.

It follows that $\langle v | w \rangle = 0$ is equivalent to $\langle w | v \rangle = 0$ unless $\langle v | v \rangle = \langle w | w \rangle = 0$. In the latter case, let $u = v + w$. If $\langle u | u \rangle = 0$ then $\langle v | w \rangle = -\langle w | v \rangle$; otherwise, $\langle v | w \rangle = 0$ iff $\langle v | u \rangle = 0$ iff $\langle u | v \rangle = 0$ iff $\langle w | v \rangle = 0$. This proves that (v) implies (i).

For non-degenerate ε -hermitian V_F , in order to prove that $\lambda \mapsto \lambda^*$ is an involution, by (15) of §1.5 we may assume $\varepsilon \pm 1$. As (7) in §1.3 follows from (9), we have $(\lambda^*)^* = \varepsilon^{-1}\lambda\varepsilon = \lambda$. The alternate case is dealt with in (12) of §1.4. \square

A sesquilinear space V_F , over a division ring F with involution, which is ε -hermitian for some ε and non-degenerate will be called *pre-hermitian*. In the sequel, we consider only pre-hermitian spaces. If V_F is, in addition, anisotropic, we also speak of an *inner product space*.

For vectors $u, v \in V$, we say that v is *orthogonal* to u and write $u \perp v$, if $\langle u | v \rangle = 0$. The *orthogonal* of $X \subseteq V$ is the subspace $X^\perp = \{v \in V \mid \forall u \in X. u \perp v\}$. A subspace U is *closed* if $U = U^{\perp\perp}$. If U is a subspace of $\dim U = 1$ then $U^\perp = \ker f$ for the linear map $f : V \rightarrow F$ given as $f(w) = \langle v | w \rangle$ where $U = vF$; f is surjective since V_f is non-degenerate, whence $\dim V/U^\perp = 1$. Since $U^\perp = \bigcap_{v \in B} vF^\perp$ for any basis B of U , it follows $\dim V/U^\perp \leq \dim U$ for any U with $\dim U < \omega$ (actually, equality holds and U is closed, see Propositions 7.1 and 5.2).

On any linear subspace U of V_F , one has the sesquilinear *subspace* U_F with the induced scalar product. When U_F is non-degenerate, U_F is pre-hermitian, too. A finite-dimensional subspace U_F of V_F is non-degenerate if and only if $U \cap U^\perp = 0$, if and only if $V = U \oplus U^\perp$ (as $\dim V/U^\perp \leq \dim U$). We write in this case $U \in \mathcal{O}(V_F)$ and say that U is a *finite-dimensional orthogonal summand*; in particular, U is closed.

Proposition 2.2. *Every pre-hermitian space V_F is the directed union of the subspaces U_F , $U \in \mathcal{O}(V_F)$. Actually, for any finite-dimensional subspace $W \in \mathbb{L}(V_F)$ there is $U \in \mathcal{O}(V_F)$ such that $W \subseteq U$ and $\dim U \leq 2 \dim W$.*

Proof. This is [20, Chapter I, §5 Lemma 4], cf. [2, Remark 4.6.14]. Alternatively, one can apply [22, Theorem 1.2] to the “orthogeometry” $\mathbb{G}(V_F)$ associated with V_F and Proposition 7.1, below. \square

For a subspace U_F of V_F , the linear subspace $\text{rad } U = U \cap U^\perp$ is the *radical* of U_F . With $\langle v + \text{rad } U \mid w + \text{rad } U \rangle := \langle v \mid w \rangle$, the F -vector space $U/\text{rad } U$ is a sesquilinear space $U_F/\text{rad } U$ with respect to the given anti-automorphism of F . We call $U_F/\text{rad } U$ a *subquotient space*.

Proposition 2.3. *Let V_F be a pre-hermitian space and let U_F be a subspace of V_F . Then $U_F/\text{rad } U$ is non-degenerate; it is ε -hermitian if V_F is. The space $U_F/\text{rad } U$ is isomorphic to any subspace W_F of V_F such that $U = W \oplus \text{rad } U$.*

Proof. The map $w \mapsto w + \text{rad } U$ establishes an isomorphism of sesquilinear spaces from W_F onto $U_F/\text{rad } U$. \square

3. RINGS AND ALGEBRAS WITH INVOLUTION

When mentioning rings, we always mean associative rings R possibly without unit. The principal right ideal aR generated by a equals $\{za \mid z \in \mathbb{Z}\} \cup \{ar \mid a \in I\}$. A $*$ -ring is a ring R endowed with an *involution*; that is, an anti-automorphism $x \mapsto x^*$ of order 2, such that

$$(r + s)^* = r^* + s^*, \quad (rs)^* = s^*r^*, \quad (r^*)^* = r \quad \text{for all } r, s \in R,$$

cf. [31, §1], [47, §2.13], [2, §4].

An element e of a $*$ -ring R is a *projection*, if $e = e^2 = e^*$. A $*$ -ring R is *proper* if $r^*r = 0$ implies $r = 0$ for all $r \in R$. Throughout this paper, let Λ be a commutative $*$ -ring with unit. A $*$ - Λ -algebra R is an associative (left) unital Λ -algebra, with unit 1 considered a constant, which is a $*$ -ring such that

$$(\lambda r)^* = \lambda^* r^* \quad \text{for all } r \in R, \lambda \in \Lambda.$$

For example, involutive Banach algebras are $*$ - \mathbb{C} -algebras. Unless stated otherwise, we consider the scalars $\lambda \in \Lambda$ as unary operations $r \mapsto \lambda r$ on R ; in other words, we consider $*$ - Λ -algebras as 1-sorted algebraic structures. The map $\lambda \mapsto \lambda 1$ is a $*$ -ring homomorphism from Λ into the center of R ; in view of this, denoting both involutions on R and on Λ by the same $*$ should not cause confusion; also, most arguments concerning the action of Λ are obvious and left to the reader.

A $*$ -ideal of a $*$ -ring or an $*$ - Λ -algebra R is an ideal I with $I = I^*$, where $I^* = \{r^* \mid r \in I\}$. We call R *strictly subdirectly irreducible* if the underlying ring is subdirectly irreducible, i.e. has a smallest non-zero ideal I ; in this case, $I = I^*$. Similarly, R is *strictly simple* if 0 and R are the only ideals. In the $*$ -ring literature, such $*$ -rings are called ‘simple’, while simple $*$ -rings are called ‘ $*$ -simple’ cf. [3].

The [right] *socle* $\text{Soc}(R)$ consists of all $a \in R$ such that aR is the sum of finitely many minimal right ideals; $\text{Soc}(R)$ is an ideal of R . We say that a $*$ - Λ -algebra is *atomic* if any non-zero right ideal contains a minimal one.

A ring R is [von Neumann] *regular* if for any $a \in R$, there is an element $x \in R$ such that $axa = a$; such an element is called a *quasi-inverse* of a . A $*$ -ring R is *$*$ -regular* if it is regular and proper. The reader interested in more details is referred to any of [45, 4, 5, 40, 16, 49].

Recall that, for a vector space V_F over a division ring F , $\text{End}(V_F)$ denote the set of all endomorphisms of V_F .

Proposition 3.1. (i) *For any vector space V_F , $\text{End}(V_F)$ is a regular simple ring.*

- (ii) *A ring R is regular if it admits a regular ideal I such that R/I is regular. Any ideal of a regular ring is regular.*
- (iii) *A ring R is regular [\ast -ring R is \ast -regular] if and only if for any $a \in R$ there is an idempotent [a (unique) projection, respectively] $e \in R$ such that $aR = eR$.*
- (iv) *For any a, b in a regular ring R , there is an idempotent $e \in aR + bR$ such that $ea = a$ and $eb = b$.*
- (v) *Homomorphic images and direct products of regular (\ast -regular) \ast - Λ -algebras are again regular (\ast -regular) \ast - Λ -algebras.*

Proof. Statements (i)-(v) are well known, cf. [5, 1.26], [16, Lemma 1.3], [16, Theorem 1.7]. For the existence of projections see [45, Part II Chapter IV Theorem 4.5] or [5, Proposition 1.13]. In (v), the claim for products is obvious, for homomorphic images it follows by (iii). \square

In particular, in the \ast -regular case, any ideal is a \ast -ideal by Proposition 3.1(iii); thus subdirectly irreducibles [simples] are strictly subdirectly irreducible [strictly simple, respectively]. Call a \ast - Λ -algebra R *primitive* if the underlying ring is primitive, that is, admits a faithful irreducible module.

Proposition 3.2. *Every regular strictly subdirectly \ast - Λ -algebra is primitive.*

Proof. Any regular ring is semi-simple, i.e. has zero radical, cf. [16, Corollary 1.2]. Hence the ring R is a subdirect product of primitive rings, cf. [33, Chapter I, §3, Theorem 1]. Being subdirectly irreducible, the ring R is therefore primitive, cf. the proof of [44, Corollary 3.4]. \square

4. ENDOMORPHISM RINGS

In the sequel, let F be a \ast - Λ -algebra where the underlying ring of F is a division ring and V_F a pre-hermitian space over F . $\text{End}(V_F)$ denotes the unital Λ -algebra of V_F of all endomorphisms of the vector space V_F . The algebra $\text{End}^\ast(V_F)$ is defined in the following Proposition which is obvious in view of Proposition 2.1.

Proposition 4.1. *The endomorphisms of V_F having an adjoint form a Λ -subalgebra $\text{End}^\ast(V_F)$ of $\text{End}(V_F)$ which is \ast - Λ -algebra with the involution $\varphi \mapsto \varphi^\ast$. If V'_F is similar to V_F then $\text{End}^\ast(V_F)$ and $\text{End}^\ast(V'_F)$ are isomorphic \ast - Λ -algebras.*

Observe that for $v \in V$, $\lambda \in \Lambda$, and $\varphi \in \text{End}^\ast(V_F)$, one has

$$(\lambda\varphi)(v) = \varphi(v)\lambda, \quad (\lambda\varphi)^\ast = \lambda^\ast\varphi^\ast.$$

Also recall the well known facts that for any $\varphi, \psi \in \text{End}^\ast(V_F)$

$$\text{im } \varphi \subseteq (\text{im } \psi)^\perp \Leftrightarrow \varphi^\ast \circ \psi = 0 \text{ and } (\text{im } \varphi)^\perp = \ker \varphi^\ast.$$

Proposition 4.2. *For any subspace U of V_F , one has $V = U \oplus U^\perp$ if and only if there is a projection $\pi_U \in \text{End}^\ast(V_F)$ such that $U = \text{im } \pi_U$. Such a projection π_U is unique.*

Projection π_U in terms of Proposition 4.2 is called the *orthogonal projection* onto U . Par abus de langage, π_U also denotes the induced epimorphism $V \rightarrow U$, while ε_U denotes the inclusion map $U \rightarrow V$. Observe that π_U and ε_U are adjoints of each other in the sense that

$$\langle \varepsilon_U(u) \mid v \rangle = \langle u \mid \pi_U(v) \rangle \quad \text{for all } u \in U, v \in V.$$

Moreover, the computational rules of $\mathbf{End}^*(V_F)$ yield, in particular, $(\varepsilon_U \varphi \pi_U)^* = \varepsilon_U \varphi^* \pi_U$ for any $\varphi \in \mathbf{End}^*(U_F)$. Finally, $\pi_U \varepsilon_U = \text{id}_U$, while $\pi_U \varepsilon_U \pi_U = \pi_U$ and $U^\perp = \ker(\varepsilon_U \pi_U)$.

Let $\dim V_F = n < \omega$. We say that bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ of V_F are a *dual pair of bases*, whenever $\langle v_i \mid w_i \rangle = 1$ for all $i \in \{1, \dots, n\}$ and $\langle v_i \mid w_j \rangle = 0$ for all $i \neq j$.

Proposition 4.3. *Let V_F be a pre-hermitian space and let $\dim V_F = n < \omega$.*

- (i) *There is a dual pair of bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ of V_F . Moreover, for any $\varphi \in \mathbf{End}(V_F)$ with $\varphi(v_j) = \sum_i w_i a_{ij}$, $\varphi^* \in \mathbf{End}(V_F)$ exists and $\varphi^*(v_i) = \sum_j w_j a_{ij}^*$. In particular, $\mathbf{End}^*(V_F)$ contains all endomorphisms of V_F and $\mathbf{End}^*(V_F)$ is regular.*
- (ii) *$\mathbf{End}^*(V_F)$ is \ast -regular if and only if V_F is anisotropic.*
- (iii) *If $U_F \in \mathbb{O}(V_F)$ then $\mathbf{End}^*(U_F) \times \mathbf{End}^*(U_F^\perp)$ embeds into $\mathbf{End}^*(V_F)$, in particular $\mathbf{End}^*(U_F)$ is a homomorphic image of a regular \ast - Λ -subalgebra of $\mathbf{End}^*(V_F)$.*

Proof. For existence of dual bases, see [33, §IV 15] or [36, §II.6]. Straightforward and well known calculations prove (i). Regularity of $\mathbf{End}^*(V_F)$ follows from Proposition 3.1(i). (ii) Assume $R = \mathbf{End}^*(V_F)$ is \ast -regular; given $v \neq 0$, choose $V = vF \oplus U$ and the endomorphism φ with $\varphi(v) = v$ and $\varphi|_U = 0$. Since R is \ast -regular, one has $\varphi R = \pi R$ for some (orthogonal) projection and $V = vF \oplus^\perp W$ for some W . If one had $\langle v \mid v \rangle = 0$, then $v \perp V$, contradicting the assumption that V_F is non-degenerate. Then converse is well known and easy to prove. In (iii), let R consist of all $\varphi \in \mathbf{End}^*(V_F)$ which leave both U and U^\perp invariant. As $R \cong \mathbf{End}^*(U_F) \times \mathbf{End}^*(U_F^\perp)$, we get (iii). \square

We put $J(V_F) = \{\varphi \in \mathbf{End}^*(V_F) \mid \dim \text{im } \varphi < \omega\}$. Cf. [2, Theorem 4.6.15] for the following.

Proposition 4.4. *Let V_F be a pre-hermitian space.*

- (i) *First of all, $J(V_F)$ is an ideal and a strictly simple regular \ast - Λ -subalgebra of $\mathbf{End}^*(V_F)$ without unit.*
- (ii) *The principal right ideals of $J(V_F)$ are in 1-1-correspondence with the finite-dimensional subspaces of V_F via the map $\varphi J(V_F) \mapsto \text{im } \varphi$; moreover, one has $\varphi_0 J(V_F) \subseteq \varphi_1 J(V_F)$ equivalent to $\text{im } \varphi_0 \subseteq \text{im } \varphi_1$.*
- (iii) *Let R be a subring of $\mathbf{End}(V)$ with $R \supseteq J(V_F)$. Then the minimal right ideals of R are of the form φR , where $\varphi \in J(V_F)$ is an idempotent such that $\varphi J(V_F)$ is a minimal right ideal of $J(V_F)$, that is, $\dim \text{im } \varphi = 1$. In particular, R is atomic and $J(V_F) = \mathbf{Soc}(R)$ is its smallest non-zero ideal.*
- (iv) *For any $\varphi_1, \dots, \varphi_n \in J(V_F)$, there is $U \in \mathbb{O}(V_F)$ such that $\pi_U \varphi_i = \varphi_i = \varphi_i \pi_U$ for all $i \in \{1, \dots, n\}$.*
- (v) *The space V_F is alternate if and only if $J(V_F)$ does not contain a projection generating a minimal right ideal. If V_F is alternate then $\pi \circ \pi^* = 0 = \pi^* \circ \pi$ for any idempotent π with $\dim \text{im } \pi = 1$.*

Proof. (i) Clearly, $J(V_F)$ is an ideal and a Λ -subalgebra of $\mathbf{End}^*(V_F)$ (without unit). Observe that $\pi_U \in J(V_F)$ for any $U \in \mathbb{O}(V_F)$ by Proposition 4.2. Moreover by Proposition 2.2, for any subspace W of V_F with $\dim W < \omega$, there exists $U \in \mathbb{O}(V_F)$ such that $W \subseteq U$.

Consider $\varphi \in J(V_F)$ and recall that the subspaces $\ker \varphi = (\operatorname{im} \varphi^*)^\perp$ and $\ker \varphi^* = (\operatorname{im} \varphi)^\perp$ are both closed. To prove that $\varphi^* \in J(V_F)$, choose $W \in \mathcal{O}(V_F)$ such that $W \supseteq \operatorname{im} \varphi = (\ker \varphi^*)^\perp$. Then $W^\perp \subseteq \ker \varphi^*$, whence $\operatorname{im} \varphi^* = \varphi^*(W)$ is finite-dimensional. It follows that

- (*) For any $\varphi_1, \dots, \varphi_n \in J(V_F)$, there is $U \in \mathcal{O}(V_F)$ such that $U \supseteq \operatorname{im} \varphi_i + \operatorname{im} \varphi_i^*$ for all $i \in \{1, \dots, n\}$ and $\varphi_i(U) = \operatorname{im} \varphi_i$ and $\varphi_i^*(U) = \operatorname{im} \varphi_i^*$. In particular,
 - (a) U is a finite-dimensional pre-hermitian space;
 - (b) $V = U \oplus U^\perp$;
 - (c) $U^\perp \subseteq \bigcap_i \ker \varphi_i \cap \ker \varphi_i^*$;
 - (d) $\pi_U \in J(V_F)$;
 - (e) $\varepsilon_U \psi \pi_U \in J(V_F)$ and $(\varepsilon_U \psi \pi_U)^* = \varepsilon_U \psi^* \pi_U$ for any $\psi \in \operatorname{End}(U_F)$.

To prove that φ has a quasi-inverse in $J(V_F)$, choose for φ a subspace $U \in \mathcal{O}(V_F)$ according to (*). By Proposition 4.3(i), $\pi_U \varphi \varepsilon_U \in \operatorname{End}^*(U_F)$ has a quasi-inverse $\psi \in \operatorname{End}^*(U_F)$. We claim that $\chi = \varepsilon_U \psi \pi_U$ is a quasi-inverse of φ in $J(V_F)$. Indeed, $\chi \in J(V_F)$ by (e) and $\varphi(v) = 0 = \chi(v)$ for any $v \in U^\perp$ by (c) and $\varphi \chi \varphi(v) = \pi_U \varphi \varepsilon_U \psi \pi_U \varphi \varepsilon_U(v) = \pi_U \varphi \varepsilon_U(v) = \varphi(v)$ for any $v \in U$.

To prove that $J(V_F)$ is strictly simple, it suffices to show that for any $0 \neq \varphi, \psi \in J(V_F)$, ψ belongs to the ideal generated by φ . Again, choose for φ and ψ a subspace $U \in \mathcal{O}(V_F)$ according to (*). Applying Proposition 3.1(i) to $\pi_U \varphi \varepsilon_U, \pi_U \psi \varepsilon_U \in \operatorname{End}(U_F)$, we get that there are $m < \omega$ and $\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_m \in \operatorname{End}(U_F)$ such that $\pi_U \psi \varepsilon_U = \sum_{i=1}^m \tau_i \pi_U \varphi \varepsilon_U \sigma_i$. Then according to (*), $\psi = \sum_{i=1}^m \varepsilon_U \tau_i \pi_U \varphi \varepsilon_U \sigma_i \pi_U$ and $\varepsilon_U \sigma_i \pi_U, \varepsilon_U \tau_i \pi_U \in J(V_F)$ for all $i \in \{1, \dots, m\}$ by (e).

(ii) We prove first that $\varphi_0 J(V_F) \subseteq \varphi_1 J(V_F)$ is equivalent to $\operatorname{im} \varphi_0 \subseteq \operatorname{im} \varphi_1$ for any $\varphi_0, \varphi_1 \in \operatorname{End}^*(V_F)$. Suppose first that $\operatorname{im} \varphi_0 \subseteq \operatorname{im} \varphi_1$ and take an arbitrary $\psi \in J(V_F)$; then $\varphi_0 \psi, \varphi_1 \psi \in J(V_F)$. Choose for $\varphi_0 \psi$ and $\varphi_1 \psi$ a subspace $U \in \mathcal{O}(V_F)$ according to (*). Then $\xi_i = \pi_U \varphi_i \psi \varepsilon_U \in \operatorname{End}(U_F)$ for any $i < 2$ and $\operatorname{im} \xi_0 \subseteq \operatorname{im} \xi_1$. As $\dim U_F < \omega$, $\xi_0 = \xi_1 \chi$ for some $\chi \in \operatorname{End}(U_F)$. According to (c), $\varphi_0 \psi(v) = \varphi_1 \psi(v) = 0$ for any $v \in U^\perp$, whence

$$\varphi_0 \psi = \pi_U \varphi_0 \psi \varepsilon_U \pi_U = \xi_0 \pi_U = \xi_1 \chi \pi_U = \pi_U \varphi_1 \psi \varepsilon_U \chi \pi_U = \varphi_1 \psi \varepsilon_U \chi \pi_U \in \varphi_1 J(V_F),$$

as $\psi \varepsilon_U \chi \pi_U \in J(V_F)$ by (e). The reverse implication is trivial by Proposition 2.2.

Besides that, for any finite-dimensional subspace W of V_F , there is $\varphi \in J(V_F)$ such that $W = \operatorname{im} \varphi$. Indeed by Proposition 2.2, there is $U \in \mathcal{O}(V_F)$ such that $W \subseteq U$, whence $W = \operatorname{im} \psi$ for some $\psi \in \operatorname{End}(U_F)$. Then $W = \operatorname{im} \varphi$ with $\varphi = \varepsilon_U \psi \pi_U \in J(V_F)$ by Proposition 4.2 and (e). This establishes the claimed 1-1-correspondence.

(iii) For any $v \neq 0$ in V there is an idempotent π_v in $J(V_F)$ such that $\operatorname{im} \pi = vF$; for such $\pi_v J(V_F) = \pi_v R$ is a minimal right ideal of both $J(V_F)$ and R . Indeed, choose W such that $V = vF \oplus W$ and let $\pi(v) = v$, $\pi|_W = 0$. Now, for any $0 \neq \varphi \in R$ one has $\varphi(v) \neq 0$ for some $v \in V$ whence $\dim \operatorname{im} \varphi \circ \pi_v = 1$ and $\pi_v R$ a minimal right ideal contained in φR ; and the minimal right ideals of R are exactly the $\pi_v R$. If $\dim \varphi = n$, then $\varphi R = \sum_{i=1}^n \pi_{v_i} R$ where v_1, \dots, v_n is a basis of $\operatorname{im} \varphi$. Thus, R is atomic with $\operatorname{Soc}(R) = J(V_F)$ contained in any non-zero ideal.

(iv) Given $\varphi_1, \dots, \varphi_n \in J(V_F)$, choose a subspace $U \in \mathcal{O}(V_F)$ according to (*). Then $\operatorname{im} \varphi_i + \operatorname{im} \varphi_i^* \subseteq U$, whence $\pi_U \varphi_i = \varphi_i$ and $\pi_U \varphi_i^* = \varphi_i^*$.

(v) If π is a projection in the $*$ -ring $J(V_F)$ then by Proposition 4.2, it is an orthogonal projection of V_F and $\langle v \mid v \rangle \neq 0$ for any $0 \neq v \in \operatorname{im} \pi$. Thus, V_F is not alternate. Conversely, assume V_F not alternate. If $\dim V_F \geq 2$, in view of Proposition 2.1 we may assume that V_F is hermitian. By Proposition 2.2, there is a non-alternate space $0 \neq U \in \mathcal{O}(V_F)$. By [20, Chapter II §2, Corollary 1], U has an

orthogonal basis. Thus $U = W \oplus W'$, where $W' \subseteq W^\perp$ and $\dim W = 1$. It follows that $W \in \mathbb{O}(V_F)$ and that π_W is a projection generating a minimal right ideal of $J(V_F)$. If $\dim V_F = 1$, then id_V will do.

Now, let V_F be alternate and π an idempotent with $\text{im } \pi = wF \neq 0$. Then for all $v \in V$, $\langle v \mid \pi^*(w) \rangle = \langle \pi(v) \mid w \rangle = \langle w\lambda \mid w \rangle = 0$ whence $\pi^*(\pi(w)) = \pi^*(w) = 0$ and $\pi^* \circ \pi = 0$. The claim for π^* follows since $\dim \text{im } \pi^* = 1$ by (*). \square

Proposition 4.5. *Any $*$ - Λ -subalgebra R of $\text{End}^*(V_F)$ extends to a $*$ - Λ -subalgebra \hat{R} of $\text{End}^*(V_F)$ such that $J(V_F)$ is the unique minimal ideal of \hat{R} . In particular, \hat{R} is strictly subdirectly irreducible and atomic with the minimal left [right] ideals being those of $J(V_F)$. Moreover, if R is regular then \hat{R} is also regular.*

Proof. The $*$ -regular case is due to [44, Proposition 3.12]. Let $\hat{R} = R + J(V_F)$. Clearly, \hat{R} is a subalgebra of $\text{End}^*(V_F)$ and $J(V_F)$ is an ideal of \hat{R} by Proposition 4.4(i). If $I \neq 0$ is a left ideal of \hat{R} then choose $\varphi \in I$ such that $\varphi \neq 0$. Then by Proposition 2.2, $0 \neq \pi_U \varphi \in J(V_F) \cap I$ for some $U \in \mathbb{O}(V_F)$. By Proposition 4.4(iii), there is a minimal left ideal $M \subseteq J(V_F) \pi_U \varphi \subseteq I$ of $J(V_F)$. Then M is also a minimal left ideal of \hat{R} . If I is an ideal of \hat{R} , then arguing as above and applying simplicity of $J(V_F)$, which follows from Proposition 4.4(i), we get that $J(V_F) \subseteq I$. Finally, Propositions 3.1(ii) and 4.4(i) imply regularity of \hat{R} when R is regular. \square

In particular, Proposition 4.5 applies to R consisting of the endomorphisms $v \mapsto v\lambda$ (also denoted as λid_V), where λ is in the center of F ; in this case, we denote the corresponding subalgebra \hat{R} by $\hat{J}(V_F)$.

A representation of a $*$ - Λ -algebra R within a per-hermitian space V_F is a homomorphism $\varepsilon : R \rightarrow \text{End}^*(V_F)$ of $*$ - Λ -algebras; it is *faithful* if ε is an injective map. In the following, existence is due to Jacobson [33, Chapter IV, §12, Theorem 2] and Kaplansky [31, Theorem 1.2.2]. Uniqueness is based on an approach via the Jacobson Density Theorem, cf. [2, Theorem 4.6.8].

Theorem 4.6. *Any primitive $*$ - Λ -algebra having a minimal right ideal is atomic with $\text{Soc}(R)$ as smallest non-zero ideal and admits a faithful representation ε within some pre-hermitian space V_F such that $\varepsilon(\text{Soc}(R)) = J(V_F)$. Up to similitude, the space V_F is uniquely determined by $\text{Soc}(R)$.*

Proof. If the underlying ring of R is a division ring then a representation is given via the scalar product $\langle \lambda \mid \mu \rangle = \lambda^* \mu$. Conversely, given a representation in V_F we have $J(V_F) \cong R$ and may assume $R = \text{End}^*(F_F)$. Up to scaling we have F with involution ν and scalar product $\langle \lambda \mid \mu \rangle = \lambda^\nu \langle 1 \mid 1 \rangle \mu = \lambda^\nu \mu$. For the endomorphism given by $\varphi_\lambda(\mu) = \lambda \mu$ one obtains $\lambda^\nu = \langle \varphi_\lambda(1) \mid 1 \rangle = \langle 1 \mid \varphi_\lambda^*(1) \rangle = \varphi_\lambda^*(1)$; that is, ν is determined by the involution on R .

Assume that R is not a division ring. First, we ignore the action of Λ . By [33, Chapter IV, §12, Theorem 1] there is a non-degenerate sesquilinear space V_F and an embedding $\varepsilon : R \rightarrow \text{End}^*(V_F)$ such that $\varepsilon(R) \supseteq J(V_F)$. Since $\dim V_F \geq 2$, Proposition 2.1 applies and V_F is pre-hermitian, cf. [33, Chapter IV, §12, Theorem 2]. The remaining claims about R follow from Proposition 4.4.

In order to discuss uniqueness of V_F as well as the action of Λ , we have a closer look on how R relates to the pre-hermitian space V_F , given a $*$ -ring embedding $\varepsilon : R \rightarrow \text{End}^*(V_F)$ such that ε maps $J = \text{Soc}(R)$ onto $J(V_F)$.

Let e be an idempotent such that eR is a minimal right ideal, $\pi = \varepsilon(e)$, $U = \text{im } \pi$, and $W = \text{im}(\text{id}_V - \pi)$; then $V = U \oplus W$ and $\dim U = 1$. Choose $0 \neq u_0 \in U$. For $\lambda \in F$, there is unique $\varphi_\lambda \in \text{End}(V_F)$ such that $\varphi_\lambda(u_0) = u_0\lambda$ and $\varphi_\lambda|_W = 0$. Then $\alpha(\lambda) = \varepsilon^{-1}(\varphi_\lambda)$ defines a ring isomorphism of F onto the subring eRe of R . Moreover, one has an α -semilinear bijection ω from V_F onto the right eRe -vector space Re ; it is given by $\omega(v) = \varepsilon^{-1}(\varphi_v)$ where $\varphi_v(u_0) = v$ and $\varphi_v|_W = 0$. The given ring embedding $\varepsilon : R \rightarrow \text{End}^*(V_F)$ can be now described by the formula $\varepsilon(r)(v) = \omega^{-1}(r\omega(v))$. Compare the proof of [2, Proposition 4.6.4].

If the action of Λ on F is still to be defined, put $\zeta\lambda = \alpha^{-1}(\zeta\alpha(\lambda))$ for $\lambda \in F$ and $\zeta \in \Lambda$. Then ε is a Λ -algebra homomorphism from R into the Λ -algebra $\text{End}(V_F)$; indeed for any $\zeta \in \Lambda$, $r \in R$, and $v \in V$ one has $\omega(v) = se$ for some $s \in R$ whence $\varepsilon(\zeta r)(v) = \omega^{-1}((\zeta r)\omega(v)) = \omega^{-1}(\zeta rse) = \omega^{-1}(rsee\zeta e) = \omega^{-1}((r\omega(v))(e\zeta e)) = \omega^{-1}((r\omega(v))\alpha^{-1}(e\zeta e)) = (\varepsilon(r)(v))\alpha^{-1}(e\zeta e)$.

Assume that e is a projection; then so is π whence $u_0 \notin u_0$ and $W = U^\perp$. In view of scaling, we may assume that $\langle u_0 | u_0 \rangle = 1$. Thus, $\varphi_\lambda^* = \varphi_{\lambda^*}$ and α is an isomorphism of $*$ -rings. Also, one obtains for all $v, w \in V$

$$\langle v | w \rangle = \langle \varphi_v(u_0) | \varphi_w(u_0) \rangle = \langle u_0 | \varphi_v^*(\varphi_w(u_0)) \rangle = \langle u_0 | u_0\lambda \rangle = \lambda$$

where $\varphi_v^*(\varphi_w(u_0)) = u_0\lambda$ for some $\lambda \in F$ since $\text{im } \varphi_v^* = W^\perp = \text{im } \pi$. From $\varphi_\lambda|_W = 0 = \varphi_w|_W$, it follows $\varphi_\lambda = \varphi_v^* \circ \varphi_w$. Summarizing, the space V_F is determined, up to scaling, by the $*$ -ring $J(V_F)$. Given another $V'_{F'}$ and $\varepsilon' : R \rightarrow \text{End}^*(V'_{F'})$, as in the Theorem, and $u'_0 \in \text{im } \varepsilon'(e)$ chosen, accordingly, we have $\alpha' : F' \rightarrow eRe$ and $\omega' : V' \rightarrow Re$ providing an isomorphism $\beta = \alpha'^{-1} \circ \alpha : F \rightarrow F'$ of division rings and a β -semilinear bijection $\omega' \circ \omega : V_F \rightarrow V'_{F'}$ which combine into an isomorphism of the sesquilinear spaces obtained from V_F and $V'_{F'}$ by scaling, thus establishing the claimed similitude.

Now, assume that R , whence $J(V_F)$, does not have any projection generating a minimal right ideal. By Proposition 4.4(v), V_F is an alternate space; in particular $\lambda = \lambda^*$ for all $\lambda \in F$, F is commutative, and $\pi \circ \pi^* = 0 = \pi^* \circ \pi$ for any idempotent π generating a minimal right ideal in $J(V_F)$. Choose $\pi = \varepsilon(e)$. It follows $\text{im } \pi \cap \text{im } \pi^* = \text{im } \pi \cap (\text{im } \pi^*)^\perp = \text{im } \pi^* \cap (\text{im } \pi)^\perp = 0$. Also, $U' := \text{im } \pi \oplus \text{im } \pi^* \in \mathcal{O}(V_F)$ and $W = \text{im } \pi^* + U'$. Thus, for any $v \in V$, $\text{im } \varphi_v^* = W^\perp = \text{im } \pi^*$. For $\psi \in \text{End}(V_F)$ we have $\psi(\text{im } \pi^*) = \text{im } \pi$ and $\ker \psi = U' + \text{im } \pi$ if and only if $\pi \circ \psi = \psi = \psi \circ \pi^*$. For any such ψ there is unique $0 \neq \mu \in F$ such that $\psi(u_1) = u_0\mu$ – and vice versa. Also, $\text{im } \psi^* = (U' + \ker \pi)^\perp = \text{im } \pi$. Choose u_1 such that $u_1F = \text{im } \pi^*$ and $\langle u_1 | u_0 \rangle = 1$. Choosing μ (and ψ), given any $v, w \in V$ one has

$$\begin{aligned} \langle v | w \rangle &= \langle \varphi_v(u_0) | \varphi_w(u_0) \rangle = \langle \varphi_v(\psi(u_1\mu)) | \varphi_w(u_0) \rangle = \\ &= \langle u_1\mu | \psi^*(\varphi_w^*(\varphi_w(u_0))) \rangle = \langle u_1 | u_0 \rangle \mu \sigma = \mu \sigma \end{aligned}$$

where $\psi^*(\varphi_w^*(\varphi_w(u_0))) = u_0\sigma$. It follows $\langle v | w \rangle = \mu\sigma$ if and only if $\psi^* \circ \varphi_v^* \circ \varphi_w = \varphi_\sigma$. Uniqueness of V_F up to similitude follows as above, with the additional choice of u'_1 and of $\mu' = \mu$. cf. [2, Proposition 4.6.6]. \square

Remark 4.7. For a primitive ring with minimal right ideal, according to Kaplansky (cf. Corollary 4.3.4 and Theorem 4.6.2 [2]) there is an idempotent e such that eR and e^*R are minimal and either $e = e^*$ or $ee^* = 0 = e^*e$. Given such, the representation of Theorem 7.6 can be directly obtained from the Jacobson Density Theorem in the context of non-empty socle cf. [2, Theorem 4.6.2]. In the first case, eRe is a $*$ -subring of R . In the second case, S consisting of the $\lambda^+ := \lambda + \lambda^*$, $\lambda \in eRe$, is a $*$ -subring of R and $(\lambda^+)^* = \lambda^+$. Moreover, $\lambda \mapsto \lambda^+$ is a ring isomorphism of eRe

onto S with inverse $\mu \mapsto e\mu$ and one has $v\lambda = v\lambda^+$ for all $v \in Re$ and λ in eRe . Thus, in both cases, $\langle v \mid w \rangle = ev^*w$ provides the required scalar product on Re .

Proposition 4.8. *Up to isomorphism, the strictly simple artinian regular $*$ - Λ -algebras R are exactly the endomorphism algebras $\text{End}^*(V_F)$, where V_F is a pre-hermitian space and $\dim V_F < \omega$. Moreover, V_F is uniquely determined by R up to similitude; V_F is anisotropic iff R is $*$ -regular.*

Proof. Having a unit, R is noetherian and $J(V_F) = \text{End}^*(V_F)$ cf. [38, §3.3.5 Proposition 3]. Thus, this follows from Theorem 4.6. \square

5. LATTICES WITH GALOIS OPERATOR

We focus on lattices with Galois operator arising in Orthogonal Geometry, cf. [20, Chapter I §9], [21], and [24, §2]. For basics on modular lattices we refer to [11, §3–4, §10, §13], alternatively [19, Chapter V §1, §5]. We consider *lattices*, L , as algebraic structures with binary operations \cdot (*meet*) and $+$ (*join*); that is, for a suitable (unique) partial order \leq , $ab = a \cdot b = \inf\{a, b\}$, $a + b = \sup\{a, b\}$. L is *modular* if

$$a \geq c \text{ implies } a(b + c) = ab + c.$$

For both concepts there is a well known equivalent definition just by equations. If L has a smallest element 0 and if $ab = 0$ then we write $a \oplus b$ instead of $a + b$.

A *sublattice* of L is a subset of L closed under meets and joins and a lattice (modular if so is L) endowed with the restrictions of these operations; for example the *intervals* $[u, v] = \{x \in L \mid u \leq x \leq v\}$. A *homomorphism* $\varphi : L \rightarrow M$ between lattices is a map such that $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in L$. A *congruence (relation)* on a lattice L is an equivalence relation θ which is compatible with meet and join, that is $a \theta b$ and $c \theta d$ jointly imply $ac \theta bd$ and $(a + c) \theta (b + d)$. If $\varphi : L \rightarrow M$ is a homomorphism then $a \theta b \Leftrightarrow \varphi(a) = \varphi(b)$ defines a congruence on L ; any congruence arises this way with surjective φ and if L is modular so is M . A lattice L is *subdirectly irreducible* if it has a smallest non-trivial congruence μ , the *monolith* of L .

A modular lattice L has *dimension* $n < \omega$, (which is denoted by $\dim L$), if L has $(n+1)$ -element maximal chains. If L has smallest element 0 , we put $\dim a = \dim[0, a]$ if that exists; we call a an *atom* if $\dim a = 1$.

A *bounded lattice* has smallest element $0 = \inf L$ and greatest element $1 = \sup L$ which are considered as constants. A bounded lattice L is *complemented* if for any $a \in L$, there is $b \in L$ such that $a \oplus b = 1$. In a CML (i.e., a complemented modular lattice) L , any interval $[u, v]$ is complemented, too; L is *atomic* if for any $a > 0$ there is an atom $p \leq a$. It follows that for elements a and b of an atomic CML L , $a \not\leq b$ in L if and only if there is an atom $p \in L$ such that $p \leq a$ and $p \not\leq b$. Primary examples of atomic CMLs are the lattices of all subspaces of vector spaces (see Proposition 6.1, below).

Lattices relevant in orthogonal geometry also support an operation $X \mapsto X^\perp$, the Galois correspondence induced by the orthogonality relation, see [20]. This is captured by the following concept (cf. [21, 24]). A *Galois lattice* is a bounded lattice L endowed with an additional operation $x \mapsto x'$ such that $x \leq y'$ implies $y \leq x'$ for any $x, y \in L$ and such that $1' = 0$. It is well known and easy to prove that $x \leq x''$, that $x \leq y$ implies $y' \leq x'$, that $x''' = x'$, that $0' = 1$, and that $(x + y)' = x'y'$. For

an equational definition see [24, IV.2.5]. A *Galois sublattice* S of a Galois lattice L is a sublattice of L such that $0, 1 \in S$ and $x' \in S$ for any $x \in S$; thus, it is a Galois lattice with the inherited operations. Similarly, a homomorphism $\varphi : L \rightarrow M$ between Galois lattices is a lattice homomorphism $L \rightarrow M$ preserving $0, 1$ and such that $\varphi(x') = (\varphi(x))'$ for all $x \in L$. Also, a lattice congruence θ on L is a *Galois lattice congruence* if $a \theta b$ implies $a' \theta b'$ for all $a, b \in L$.

A *polarity lattice* is a Galois lattice such that p' is a dual atom of L for any atom $p \in L$. The referee pointed out the need for such concept and provided the smallest example of a Galois CML which is not a polarity lattice: the 4-element CML with $x' = 0$ for $x \neq 0$ and $0' = 1$. Also, the class of modular polarity lattices is not closed under substructures: Consider a subspace X of a Hilbert space H such that $X \neq X^c$, the closure of X ; then then $0, X, X^c, X^\perp, X+X^\perp, H$ form a subalgebra of the polarity lattice associated with H cf. Proposition 7.1 below. It remains open whether the class of CML polarity lattices is closed under complemented subalgebras.

A Galois lattice L is a *lattice with involution* if, in addition, $x'' = x$ for all $x \in L$; equivalently, if $x \mapsto x'$ is a dual automorphism of order 2 of the lattice L ; in particular, such is a polarity lattice. L is an *ortholattice* if, in addition, the involution satisfies the identity $xx' = 0$ (or equivalently, $x + x' = 1$). We write MIL [CMIL] shortly for a [complemented] modular lattice with involution and MOL for a modular ortholattice. We use each abbreviation also to denote the class of *all* Galois lattices with the corresponding property. Observe that $\dim u = \dim[u', 1]$ in any MIL. Also, observe that in a CMIL, in general, x' fails to be a complement of x : cf. the 4 element CML with $p = p'$ for each atom. The following statement is straightforward to prove.

Lemma 5.1. *Let L_0, L_1 be lattices with involution.*

- (i) *A map $\varphi : L_0 \rightarrow L_1$ is a homomorphism, if $\varphi(x+y) = \varphi(x) + \varphi(y)$, $\varphi(x') = \varphi(x)'$ for all $x, y \in L_0$, and $\varphi(0) = 0$.*
- (ii) *A subset $X \subseteq L_0$ is a Galois sublattice of L_0 , if $0 \in X$ and X is closed under the operations $+$ and $'$.*

For a modular polarity lattice L , let $L_f = F \cup \{u' \mid u \in F\}$ where $F = \{u \in L \mid \dim u < \omega\}$.

Proposition 5.2. *If L is a polarity CML then L_f is a Galois sublattice of L and L_f is an atomic CMIL which is the directed union of its subalgebras $[0, u] \cup [u', 1]$, where $\dim u < \omega$ and $u \oplus u' = 1$ (which are all CMILs). If L is a CMIL, then $L_f = \{a \in L \mid \dim a < \omega \text{ or } \dim[a, 1] < \omega\}$.*

Proof. We have $\dim[v', u'] \leq \dim[u, v]$ if the latter is finite. Indeed, if $\dim[u, v] = 1$ then $v = u + p$, where p is a complement of u in $[0, v]$, whence an atom and so $v' = u'p'$ is a lower cover of u' unless $v' = u'$. Thus, if $\dim u < \omega$, then $\dim u'' \leq \dim[u', 1] \leq \dim u$, $u'' = u$, and $x \mapsto x'$ provides a pair of mutually inverse lattice anti-isomorphisms between the intervals $[0, u]$ and $[u', 1]$ of L . Therefore, $[u', 1] \subseteq L_f$. Since $\{u \in L \mid \dim u < \omega\}$ is closed under joins and $0 \in L_f$, L_f is a Galois sublattice of L by Lemma 5.1(ii) and, in particular, L_f is a MIL.

If $X \subseteq L_f$ is finite, then there is $u \in L$ such that $\dim u < \omega$ and $X = Y \cup Z$, where $y, z' \in [0, u]$ for all $y \in Y, z \in Z$. Choose v as a complement of $u + u'$ in $[u, 1]$. Then $\dim[u, v] = \dim[u + u', 1] \leq \dim[u', 1] = \dim u < \omega$ whence $\dim v < \omega$. We have $u, u', v \in L_f$. Therefore, $u = v(u + u')$ implies $u' = v' + uu'$. It follows

that $v + v' = v + u + uu' + v' = v + u + u' = 1$ and $vv' = (v + v')' = 1' = 0$ whence $X \subseteq [0, v] \cup [v', 1]$. This proves the first statement.

If L is a CMIL and $\dim[a, 1] < \omega$ then $\dim a' = \dim[a, 1] < \omega$, thus $a' \in L_f$ and $a = a'' \in L_f$. \square

From a lattice congruence θ on a MIL L , we put $a \theta' b :\Leftrightarrow a \theta b$. Then θ' is also a lattice congruence on L and the Galois lattice congruences on L are exactly the lattice congruences θ on L such that $\theta = \theta'$. We call an MIL L *strictly subdirectly irreducible* if the underlying lattice is subdirectly irreducible; in that case, one has $\mu = \mu'$ for the lattice monolith. Similarly, the MIL L is *strictly simple* if the underlying lattice is simple. In the case of MOLs one has $\theta = \theta'$ for all θ ; thus subdirectly irreducible MOLs [simple MOLs] are strictly subdirectly irreducible [strictly simple, respectively]. The following is well known.

Proposition 5.3. *A subdirectly irreducible CML L is atomic provided it contains an atom. If L is, in addition, a CMIL with lattice monolith μ , then one has $a \in L_f$ iff $a \mu 0$ or $a \mu 1$. In particular, L_f is strictly subdirectly irreducible and atomic, too.*

Proof. Let p be an atom in L . By modularity, the smallest lattice congruence μ such that $0 \mu p$ is minimal. Thus given $a > 0$, one has $0 \theta p$ in the smallest lattice congruence such that $0 \theta a$, whence by modularity, the quotient $p/0$ is projective to some subquotient c/d of $a/0$. Then any complement q of d in $[0, c]$ is an atom. Thus L is atomic and it follows that $x \mu y$ iff $\dim[xy, x + y] < \omega$. In view of Proposition 5.2, we are done. \square

The following Proposition associates a CMIL $\mathbb{L}(R)$ with any regular $*$ - Λ -algebra R .

Proposition 5.4. (i) *The principal right ideals of a regular ring R , possibly without unit, form a sublattice $L(R)$, containing 0, of the lattice of all right ideals of R ; $L(R)$ is sectionally complemented and modular. In the case with unit, $L(R)$ is a CML with top element R .*
 (ii) *For any regular $[*$ -regular] $*$ - Λ -algebra R , the CML $L(R)$ becomes a CMIL [MOL, respectively] endowed with the involution $eR \mapsto (1 - e^*)R$, where e is an idempotent [a projection, respectively]; we denote it by $\mathbb{L}(R)$.*
 (iii) *For any regular $[*$ -regular] $*$ - Λ -algebras R_i , $i \in I$, and $R = \prod_{i \in I} R_i$ one has $\mathbb{L}(R) \cong \prod_{i \in I} \mathbb{L}(R_i)$.*
 (iv) *If $\varepsilon: R \rightarrow S$ is a homomorphism and R, S are regular rings, then $\bar{\varepsilon}: L(R) \rightarrow L(S)$, $\bar{\varepsilon}: aR \mapsto \varepsilon(a)S$ is a lattice homomorphism preserving 0 and 1. If ε is injective, then so is $\bar{\varepsilon}$; if ε is surjective, then so is $\bar{\varepsilon}$. If R and S are regular $*$ - Λ -algebras and ε a homomorphism of such, then $\bar{\varepsilon}: \mathbb{L}(R) \rightarrow \mathbb{L}(S)$ is a homomorphism between MILs.*

In Proposition 5.4(ii), one can consider the preorder $e \leq f$ iff $fe = e$ on the set of idempotents of R and obtain the lattice $\mathbb{L}(R)$ factoring by the equivalence relation $e \sim f$ iff $e \leq f \leq e$; the involution is given by $e \mapsto 1 - e^*$. In the $*$ -regular case, any of the equivalence classes contains a unique projection so that $\mathbb{L}(R)$ is also called the *projection [ortho]lattice* of R .

Recalling Proposition 4.4 for a pre-hermitian space V_F , the principal right ideals of $J(V_F)$ form an atomic sectionally complemented sublattice of the lattice of all right ideals of $J(V_F)$, which is isomorphic to the lattice of finite-dimensional subspaces of V_F via the map $\varphi J(V_F) \mapsto \text{im } \varphi$.

Proof. These results originate with [13]. (i)–(ii): See [51, §8-3.3.13]. For $R \in \mathcal{R}_\Lambda$, the map $eR \mapsto R(1-e) \mapsto (1-e^*)R$ combines a dual isomorphism of $L(R)$ onto the lattice of left principal ideals with an isomorphism of the latter onto $L(R)$. (iii)–(iv): See [51, §8-3.3.14–15]. \square

6. PROJECTIVE SPACES AND ORTHOGEOMETRIES

Synthetic geometries are a convenient link between lattice and vector space structures. We follow [12, Chapter 2]. A *projective space* P is a set, whose elements are called *points*, endowed with a ternary relation $\Delta \subseteq P^3$ of *collinearity* satisfying the following conditions:

- (i) if $\Delta(p_0, p_1, p_2)$, then $\Delta(p_{\sigma(0)}, p_{\sigma(1)}, p_{\sigma(2)})$ and $p_{\sigma(0)} \neq p_{\sigma(1)}$ for any permutation σ on the set $\{0, 1, 2\}$;
- (ii) if $\Delta(p_0, p_1, a)$ and $\Delta(p_0, p_1, b)$, then $\Delta(p_0, a, b)$;
- (iii) if $\Delta(p, a, b)$ and $\Delta(p, c, d)$, then $\Delta(q, a, c)$ and $\Delta(q, b, d)$ for some $q \in P$.

The space P is *irreducible* if for any $p \neq q$ in P there is $r \in P$ such that $\Delta(p, q, r)$. A set $X \subseteq P$ is a *subspace* of P if $p, q \in X$ and $\Delta(p, q, r)$ together imply that $r \in X$. Any projective space P is the disjoint union of its irreducible subspaces P_i , $i \in I$, which are called its *components*. The set $L(P)$ of all subspaces of an [irreducible] projective space P is a complete [subdirectly irreducible] atomic CML, in which the atoms are the subspaces $\{p\}$, $p \in P$, all of which are compact. Moreover, $L(P) \cong \prod_{i \in I} L(P_i)$ via the map $X \mapsto (X \cap P_i \mid i \in I)$. Conversely, any complete atomic CML L with compact atoms is isomorphic to $L(P)$ via the map $a \mapsto \{p \in P \mid p \leq a\}$, where P is the set of atoms of L and distinct points $p, q, r \in P$ are collinear if and only if $r < p + q$. Recall that Jónsson's *Arguesian* lattice identity [34] holds in $L(P)$ if and only if P is desarguean. For a vector space V_F , let $L(V_F)$ denote the lattice of all linear subspaces of V_F .

Proposition 6.1. (i) For any vector space V_F , $L(V_F)$ is a CML. Moreover, there exists an irreducible desarguean projective space P such that $L(V_F) \cong L(P)$.
(ii) For any irreducible desarguean projective space P with $\dim L(P) > 2$, there is a vector space V_F such that $L(P) \cong L(V_F)$; F is unique up to isomorphism, V_F up to semilinear bijection.
(iii) If P is irreducible and $\dim L(P) > 3$, then P is desarguean.
(iv) Any subdirectly irreducible CML of dimension at least 4 is Arguesian.

Proof. Claim (i) is the content of [12, Proposition 2.4.15]. For (ii), see [12, Proposition 2.5.6] and [12, Chapter 9]. For (iii), see [11, Chapter 13]. As to claim (iv), according to Frink [14], any CML L embeds into $L(P)$ for some projective space P . Since L is subdirectly irreducible as a lattice, it embeds into $L(P_i)$ for some irreducible component P_i of P , which is desarguean since $\dim L(P_i) > 3$, whence statement (iv) follows. \square

Proposition 6.2. Let P be a projective space. There is a 1-1-correspondence between maps $X \mapsto X^\perp$ turning $L(P)$ into a Galois CML $\mathbb{L}(P, \perp)$ on one side and, on the other side, symmetric binary relations \perp on P such that

- (a) for any $p, q, r, s \in P$, if $p \perp q$, $p \perp r$, and $\Delta(q, r, s)$ then $p \perp s$;
- (b) for any $p \in P$ there is $q \in P$ with $p \not\perp q$.

The correspondence is given by

$$X^\perp := \{q \in P \mid q \perp p \text{ for all } p \in X\}, \quad p \perp q :\Leftrightarrow p \leq q^\perp.$$

Given such relation \perp , the following are equivalent

- (i) The Galois CML $\mathbb{L}(P, \perp)$ is a polarity CML;
- (ii) for all $p, q, r \in P$, if $p \neq q$, then $r \perp t$ for some $t \in P$ such that $\Delta(p, q, t)$;
- (iii) same as (ii) with additional hypotheses $r \not\leq p$ and $r \not\leq q$.

A pair (P, \perp) satisfying all of the above is an *orthogeometry*. Compare [12, Definition 14.1.1] and [22, §4]. $\mathbb{L}(P, \perp)$ is defined, accordingly.

Proof. Given \perp , it follows $X^\perp \in L(P)$ by (a) while (b) and symmetry of \perp yield that $\mathbb{L}(P, \perp)$ is a Galois lattice. The converse is obvious. Assuming (i), if $\dim X = 2$ and $r \in P$ then $X \cap \{r\}^\perp \neq \emptyset$ by modularity, proving (ii) cf. [12, Remark 14.1.2]. While (iii) is a special case of (ii) it implies that $\{r\}^\perp$ is a coatom: by (b) choose $p \in P$, $p \not\leq r$. Now, for any $q \in P$, if $q \neq p$ and $q \notin \{r\}^\perp$ one has $\Delta(p, q, t)$ for some $t \in \{r\}^\perp$ whence $q \in \{r\} + \{r\}^\perp$. This proves $\{r\} + \{r\}^\perp = P$ and, by modularity, that $\{r\}^\perp$ is a coatom of $L(P)$. \square

For a MIL, L , let P_L denote the set of atoms of L . We define a collinearity on P_L by putting $\Delta(p, q, r)$ for distinct atoms $p, q, r \in P_L$ such that $p \leq q + r$ in L . Furthermore, we put $p \perp q$ if $p \leq q'$.

Proposition 6.3. [22, Lemma 4.2] *For any MIL L , $\mathbb{G}(L) = (P_L, \perp)$ is an orthogeometry.*

Proposition 6.4. *For any orthogeometry (P, \perp) , the Galois lattice $L = \mathbb{L}(P, \perp)_f$, consisting of all X, X^\perp with $X \in L(P)$ and $\dim X < \omega$, is a CMIL with $L = L_f$. Conversely, for any CMIL L with $L = L_f$, one has $L \cong \mathbb{L}(\mathbb{G}(L))_f$.*

Proof. See [22, Theorem 1.1] and Proposition 5.2. \square

7. SUBSPACE LATTICES

The following Proposition relates any pre-hermitian space V_F with an orthogeometry $\mathbb{G}(V_F)$ and a polarity lattice $\mathbb{L}(V_F)$.

Proposition 7.1. *If V_F is a pre-hermitian space then the CML of all linear subspaces becomes a polarity lattice $\mathbb{L}(V_F)$ with the unary operation $U \mapsto U^\perp$; moreover, $U \mapsto \{vF \mid 0 \neq v \in U\}$ defines an isomorphism of $\mathbb{L}(V_F)$ onto $\mathbb{L}(\mathbb{G}(V_F))$ where $\mathbb{G}(V_F) = (P, \perp)$ is the orthogeometry (P, \perp) with $P = \{vF \mid 0 \neq v \in V\}$ and $vF \perp wF \Leftrightarrow v \perp w$.*

Proof. Since orthogonals are subspaces, (a) of Proposition 6.2 is satisfied while (b) follows from non-degeneracy. Thus, $\mathbb{L}(V_F)$ is a Galois CML. Observe that for any $u \in V$, $f_u(v) = \langle u \mid v \rangle$ is a linear map $V_F \rightarrow F_F$; thus, one has $uF^\perp = \ker f_u$ a coatom and $\mathbb{L}(V_F)$ a polarity lattice. The isomorphism $\mathbb{L}(V_F) \rightarrow \mathbb{L}(\mathbb{G}(V_F))$ being obvious, it follows that $\mathbb{G}(V_F)$ is an orthogeometry. See also [12, Proposition 14.1.6], \square

Proposition 7.2. *Let V_F be a pre-hermitian space.*

- (i) *The polarity lattice $\mathbb{L}(V_F)_f$ is the directed union of its Galois sublattices $[0, U] \cup [U^\perp, V]$, $U \in \mathcal{O}(V_F)$. Moreover, for any $U \in \mathcal{O}(V_F)$ there is a Galois lattice embedding of $[0, U] \cup [U^\perp, V]$ into $\mathbb{L}(W_F)$ for some $W \in \mathcal{O}(V_F)$.*

- (ii) *The Galois lattices $\mathbb{L}(\mathbb{G}(V_F))_{\mathfrak{f}}$ and $\mathbb{L}(V_F)_{\mathfrak{f}}$ are mutually isomorphic and strictly subdirectly irreducible Arguesian CMILs; if V_F is anisotropic, then $\mathbb{L}(V_F)_{\mathfrak{f}}$ is a MOL.*
- (iii) *For any strictly subdirectly irreducible Arguesian CMIL L of dimension at least 3 such that $L = L_{\mathfrak{f}}$, there is a (unique up to similitude) pre-hermitian space V_F such that $L \cong \mathbb{L}(V_F)_{\mathfrak{f}}$; if L is a MOL, then V_F is anisotropic.*
- (iv) *If $\dim V_F < \omega$ then $\mathbb{L}(V_F)$ is a MIL.*

In (iv), for cardinality reasons, $\dim V_F < \omega$ is also necessary; see also [37].

Proof. The first claim in (i) and, in case $U \neq V$, $[0, U] \cup [U^{\perp}, V] \cong \mathbb{L}(U_F) \times \mathbf{2}$ follow from Propositions 5.2 and 7.1. Moreover, this Galois lattice is isomorphic to the Galois sublattice $[0, U] \cup [U^{\perp} \cap W, W]$ of $\mathbb{L}(W_F)$, choosing W by Proposition 2.2 such that $U \subset W \in \mathbb{O}(V_F)$.

To prove (ii), we notice first that $\mathbb{L}(\mathbb{G}(V_F))_{\mathfrak{f}}$ is a CMIL by Propositions 7.1 and 6.4. Moreover, as a sublattice of $\mathbb{L}(V_F)$, $\mathbb{L}(V_F)_{\mathfrak{f}}$ is an Arguesian lattice. Strict subdirect irreducibility of $\mathbb{L}(\mathbb{G}(V_F))_{\mathfrak{f}}$ follows from Fact 7.1, [12, Example 2.7.2], and [22, Corollary 1.5]. Furthermore, if V_F is anisotropic, then X^{\perp} is an orthocomplement of X for any $X \in \mathbb{L}(V_F)$ with $\dim X < \omega$.

We prove now (iii). By [22, Corollary 1.5], there is an irreducible orthogeometry (P, \perp) such that $L \cong \mathbb{L}(P, \perp)_{\mathfrak{f}}$. Combining Proposition 6.1(ii) and [12, Theorem 14.1.8], one gets a pre-hermitian space V_F such that $\mathbb{L}(P, \perp)_{\mathfrak{f}} \cong \mathbb{L}(V_F)_{\mathfrak{f}}$. For uniqueness, see [12, Theorem 14.3.4] or [20, p. 33]. If L is a MOL, then V_F is obviously anisotropic.

Finally, if $\dim V_F < \omega$, then $\mathbb{L}(V_F) = \mathbb{L}(V_F)_{\mathfrak{f}}$ is a MIL by Propositions 7.1 and 5.2. \square

Proposition 7.3. *Any Galois sublattice L of $\mathbb{L}(V_F)$ which is a MIL extends to a Galois sublattice \hat{L} of $\mathbb{L}(V_F)$ which is a MIL and such that $\hat{L}_{\mathfrak{f}} = \mathbb{L}(V_F)_{\mathfrak{f}}$. In particular, \hat{L} is a strictly subdirectly irreducible atomic MIL. Moreover, if L is a CMIL then \hat{L} is a CMIL.*

Proof. Existence of \hat{L} with the required properties follows from the proof of [22, Theorem 2.1]. In particular, \hat{L} is atomic. Strict subdirect irreducibility of \hat{L} follows from [22, Corollary 1.5], see also Proposition 7.2(ii). For a first such construction see [8]. \square

A *representation* of a MIL (or CMIL) L in V_F is a homomorphism $\varepsilon: L \rightarrow \mathbb{L}(V_F)$ of Galois lattices. It is *faithful* if it is injective, i.e. an embedding; in this case, we usually identify L with its image in $\mathbb{L}(V_F)$. A map $\varepsilon: L \rightarrow \mathbb{L}(V_F)$ is a representation if and only if it preserves joins, involution, and the least element.

Lemma 7.4. *Let ε be a representation of a MIL L in a pre-hermitian space V_F .*

- (i) *Any element in the image of ε is closed.*
- (ii) *If ε is faithful and V_F is anisotropic, then L is a MOL.*

Proof. Let $x \in L$ be arbitrary.

- (i) We have $\varepsilon(x) = \varepsilon(x'') = \varepsilon(x')^{\perp} = \varepsilon(x)^{\perp\perp}$.
- (ii) If V_F is anisotropic, then we have $\varepsilon(xx') = \varepsilon(x) \cap \varepsilon(x)^{\perp} = 0$. As ε is faithful, we conclude that $xx' = 0$. Hence $'$ is an orthocomplementation. \square

A representation of a MIL L within an orthogeometry (P, \perp) is a homomorphism $\eta: L \rightarrow \mathbb{L}(P, \perp)$. The following obvious fact relates the two concepts of a representation.

Proposition 7.5. *For a MIL L , ε is a [faithful] representation in V_F if and only if the mapping $\eta: a \mapsto \{p \in P_V \mid p \subseteq \varepsilon(a)\}$ is a [faithful] representation of L in the orthogeometry $\mathbb{G}(V_F)$.*

Theorem 7.6. *Let L be an Arguesian strictly subdirectly irreducible CMIL [MOL] such that $\dim L > 2$ and L has an atom. Then L admits a faithful representation ε within some [anisotropic] pre-hermitian space V_F such that ε induces a bijection between the sets of atoms of L and of $\mathbb{L}(V_F)$. In particular, ε restricts to an isomorphism from L_f onto $\mathbb{L}(V_F)_f$. The space V_F is unique up to similitude.*

Proof. By Proposition 5.3, L is atomic and L_f is strictly subdirectly irreducible and atomic. Moreover by Proposition 7.2(iii), $L_f \cong \mathbb{L}(V_F)_f$ for some [anisotropic] pre-hermitian space V_F which is unique up to isomorphism and scaling. By definition and Proposition 6.4, $\mathbb{G}(L) = \mathbb{G}(L_f) \cong \mathbb{G}(V_F)$. By [22, Lemma 10.4], L has a faithful representation within the orthogeometry $\mathbb{G}(L)$, whence in the orthogeometry $\mathbb{G}(V_F)$. The desired conclusion follows from Proposition 7.5. \square

The following fact is a corollary of Theorem 7.6 which is in principle already in [7].

Proposition 7.7. *Up to isomorphism, the strictly simple Arguesian CMILs L of finite dimension $n > 2$ are the $\mathbb{L}(V_F)$, where V_F is a pre-hermitian space with $\dim V_F = n$. The space V_F is determined by L up to similitude; V_F is anisotropic, iff L is a MOL.*

Proposition 7.8.

- (i) *If ε is a faithful representation of the regular $*$ - Λ -algebra R in a pre-hermitian space V_F , then the map $\eta: aR \mapsto \text{im } \varepsilon(a)$ defines a faithful representation of $\mathbb{L}(R)$ in V_F .*
- (ii) *If $\dim V_F < \omega$ then $\mathbb{L}(V_F) \cong \mathbb{L}(\text{End}^*(V_F))$.*

Proof. (i) We refer to [15]. We may assume that $R \subseteq \text{End}^*(V_F)$; that is, ε is the inclusion map. By Propositions 3.1(i) and 5.4(iv), η is a 0 and 1 preserving lattice embedding of $L(R)$ into $L(V_F)$. Moreover, for any $v \in V$ and an idempotent $\varphi \in R$, one has $v \in (\eta(\varphi R))^\perp = (\text{im } \varphi)^\perp$ iff $\langle \varphi^*(v) \mid w \rangle = \langle v \mid \varphi(w) \rangle = 0$ for all $w \in V$, iff $\varphi^*(v) = 0$, iff $v = (\text{id}_V - \varphi^*)(v)$, iff $v \in \text{im}(\text{id}_V - \varphi^*) = \eta((\varphi R)')$, whence η preserves the involution.

(ii) By (i) and Proposition 4.3(ii), the identical map ε on $\text{End}^*(V_F)$ defines a faithful representation of $\mathbb{L}(V_F)$. It is surjective since any subspace is the image of some endomorphism $\varphi \in \text{End}^*(V_F)$, cf. also Proposition 5.4(iv). \square

8. REPRESENTATIONS AS MULTI-SORTED STRUCTURES

Given the commutative $*$ -ring Λ , let \mathcal{F}_Λ denote the class of all division rings with involution which are $*$ - Λ -algebras. Unless stated otherwise, pre-hermitian spaces V_F , $F \in \mathcal{F}_\Lambda$, are dealt with as 2-sorted structures with sorts V and F . That is, V carries the structure of an abelian group, F that of a ring with involution ν and with an unary operation $\lambda \mapsto \zeta\lambda$ associated to each $\zeta \in \Lambda$; moreover one has the maps $V \times F \rightarrow V$ with $(v, \lambda) \mapsto v\lambda$ and $V \times V \rightarrow F$ with $(v, w) \mapsto \langle v \mid w \rangle$.

In general, a *similarity type* for n -sorted algebraic structures will have a list S_1, \dots, S_n of names for sorts, a list of typed operation symbols $f : S_{j_1} \times \dots \times S_{j_{k_f}} \rightarrow S_{j_{k_f+1}}$, and a list of typed relation symbols $R \subseteq S_{j_1} \times \dots \times S_{j_{k_R}}$. A *structure* A of this type is a family S_1^A, \dots, S_n^A of sets together with a map $f^A : S_{j_1}^A \times \dots \times S_{j_{k_f}}^A \rightarrow S_{j_{k_f+1}}^A$ for each operation symbol f and with a set $R^A \subseteq S_{j_1}^A \times \dots \times S_{j_{k_R}}^A$ for each relation symbol R .

Recall the notion of *ultrafilter* on a set I : a set \mathcal{U} of subsets of I which is maximal with the following properties: $\emptyset \notin \mathcal{U}$, $U \cap V \in \mathcal{U}$ for all $U, V \in \mathcal{U}$, $V \in \mathcal{U}$ for all $U \in \mathcal{U}$; in particular, either $U \in \mathcal{U}$ or $I \setminus U \in \mathcal{U}$ for any $U \subseteq I$. Given n -sorted structures A_i , $i \in I$, of fixed sorted similarity type and any ultrafilter \mathcal{U} on I , for each sort S_j one has an equivalence relation \equiv_{S_j} on the direct product $\prod_{i \in I} S_j^{A_i}$ of sets such that

$$(a_i \mid i \in I) \equiv_{S_j} (b_i \mid i \in I) \Leftrightarrow \exists U \in \mathcal{U} \forall i \in U \ a_i = b_i.$$

The equivalence classes $[a_i \mid i \in I]_{S_j}$ are the elements of the ultraproduct $S_j^A = \prod_{i \in I} S_j^{A_i} / \mathcal{U}$ of sort S_j . Now, one defines the relations and operations of the *ultraproduct* $A = \prod_{i \in I} A_i / \mathcal{U}$ as follows:

$$([a_i^{j_1} \mid i \in I]_{S_{j_1}}, \dots, [a_i^{j_{k_f}} \mid i \in I]_{S_{j_{k_f}}}) \in R^A \Leftrightarrow \exists U \in \mathcal{U} \forall i \in U \ (a_i^{j_1}, \dots, a_i^{j_{k_R}}) \in R^{A_i}$$

for each relation symbol R (with type as above)

$$f^A([a_i^{j_1} \mid i \in I]_{S_{j_1}}, \dots, [a_i^{j_{k_f}} \mid i \in I]_{S_{j_{k_f}}}) = [f^{A_i}(a_i^{j_1}, \dots, a_i^{j_{k_f}}) \mid i \in I]_{S_{j_{k_f+1}}}$$

for each operation symbol f (with type as above). The operations are well defined as one easily sees.

Proposition 8.1. *Let $\Phi(x_1, \dots, x_m)$ any formula in the first order language associated to the given similarity type with free variables x_k of sort S_{j_k} . For an ultraproduct as above one has $\Phi([a_i^{j_1} \mid i \in I]_{S_{j_1}}, \dots, [a_i^{j_m} \mid i \in I]_{S_{j_{k_m}}})$ valid in A if and only if there is $U \in \mathcal{U}$ such that $\Phi(a_i^{j_1}, \dots, a_i^{j_{k_m}})$ is valid in A_i for all $i \in U$.*

This is a variant of the well known Theorem of Łoś [32, Theorem 9.5.1]. To derive it from the 1-sorted case, multi-sorted structures may be conceived as 1-sorted relational structures, assuming sorts to be pairwise disjoint and captured by unary predicates. The following is an immediate consequence.

Lemma 8.2. *Let \mathcal{U} be any ultrafilter over a set I . Let also $(V_i)_{F_i}$ be a pre-hermitian space over $F_i \in \mathcal{F}_\Lambda$ for all $i \in I$. Then $F = \prod_{i \in I} F_i / \mathcal{U} \in \mathcal{F}_\Lambda$ and $V = \prod_{i \in I} V_i / \mathcal{U}$ is a pre-hermitian space over F . Here, for $v = [v_i \mid i \in I]$ and $w = [w_i \mid i \in I]$ in the abelian group V and $\lambda = [\lambda_i \mid i \in I]$ in F one has*

$$v\lambda = [v_i\lambda_i \mid i \in I], \quad \langle v \mid w \rangle = [\langle v_i \mid w_i \rangle_i \mid i \in I]$$

where $\langle v_i \mid w_i \rangle_i \in F_i$ is the value under the scalar product from $(V_i)_{F_i}$.

Recall that a representation of a $*$ - Λ -algebra R within a pre-hermitian space V_F is a $*$ - Λ -algebra homomorphism $\varepsilon : R \rightarrow \mathbf{End}^*(V_F)$. It is convenient to consider representations as unitary R - F -bimodules. More precisely, one has an action $(r, v) \mapsto rv = \varepsilon(r)(v)$ of R on the left and an action $(v, \lambda) \mapsto v\lambda$ of F on the right satisfying the laws of unitary left and right modules and such that

$$(\lambda r)v = (rv)\lambda = r(v\lambda) \quad \text{for all } v \in V, \ r \in R, \ \lambda \in \Lambda,$$

where $v\lambda = v(\lambda 1_F)$. Moreover,

$$\begin{aligned} \langle rx \mid y \rangle &= \langle x \mid r^*y \rangle \quad \text{for all } r \in R, x, y \in V \\ (\lambda r)^*v &= (\lambda^*r^*)v = (r^*v)\lambda^* \quad \text{for all } v \in V, r \in R, \lambda \in \Lambda. \end{aligned}$$

We denote a representation of R in V_F by ${}_R V_F$. The R - F -bimodule ${}_R V_F$ with scalar product will be considered as a 3-sorted structure with sorts V , R , and F ; the $*$ - Λ -algebras R and F are considered as 1-sorted structures, where $\lambda \in \Lambda$ serves to denote the unary operation $x \mapsto \lambda x$. Our main concern will be faithful representations; that is, representations ${}_R V_F$ such that $rv = 0$ for all $v \in V$ if only if $r = 0$. Observe that a regular algebra R is $*$ -regular, if it admits a faithful representation in an anisotropic space.

The following is as obvious as crucial: A representation of a MIL $\varepsilon: L \rightarrow \mathbb{L}(V_F)$ can be viewed as a 3-sorted structure with sorts L , V , and F and with the map ε being captured by the binary relation (cf. [42, 41, 48] for this method)

$$\{(a, v) \mid v \in \varepsilon(a)\} \subseteq L \times V,$$

which we denote by ε again.

Lemma 8.3. *Under the hypotheses of Lemma 8.2 one has the following.*

- (i) *If L_i is a MIL and $(L_i, V_i, F_i; \varepsilon_i)$ is a faithful representation for all $i \in I$, then the associated ultraproduct $(L, V_F, F; \varepsilon)$ is a faithful representation of $L = \prod_{i \in I} L_i / \mathcal{U}$.*
- (ii) *If R_i is a $*$ - Λ -algebra and ${}_R V_i$ a faithful representation for all $i \in I$, then the associated ultraproduct ${}_R V_F$ is a faithful representation of $R = \prod_{i \in I} R_i / \mathcal{U}$.*
- (iii) *Let U be an n -dimensional subspace of V_F , $n < \omega$. Then there are $J \in \mathcal{U}$ and n -dimensional subspaces U_i of $(V_i)_{F_i}$, $i \in J$, such that $U \cong \prod_{i \in J} U_i / \mathcal{U}_J$, where $\mathcal{U}_J = \{X \in \mathcal{U} \mid X \subseteq J\}$, and*

$$\mathbb{L}(U_F) \cong \prod_{i \in J} \mathbb{L}((U_i)_{F_i}) / \mathcal{U}_J, \quad \text{End}^*(U_F) \cong \prod_{i \in J} \text{End}^*((U_i)_{F_i}) / \mathcal{U}_J$$

Proof. Statements (i) and (ii) are immediate by Proposition 8.1 and the observation that both types of 3-sorted structures can be characterized by first order axioms. In (iii) observe that for a fixed positive integer n , there is a first order formula in the two sorted language for vector spaces) expressing that a set of vectors $\{v_1, \dots, v_n\}$ is independent [is a basis], as well as a first order formula expressing that a vector v is in the span of $\{v_1, \dots, v_n\}$. Thus, by the Łoś Theorem, a basis of U determines J and bases of spaces U_i , $i \in J$. Now, apply (i) to lattices $L_i = \mathbb{L}((U_i)_{F_i})$, $i \in J$, to get an embedding of $\prod_{i \in J} L_i / \mathcal{U}_J$ into $\mathbb{L}(U_F)$. Surjectivity of this embedding is granted by the sentence stating that for any v_1, \dots, v_n , there is a such that $v \in \varepsilon(a)$ if and only if v in the span of v_1, \dots, v_n . Similarly, we apply (ii) in the ring case and use the sentence stating that for any basis v_1, \dots, v_n and any w_1, \dots, w_n , there is r such that $rv_i = w_i$ for all $i \in \{1, \dots, n\}$. \square

Inheritance of existence of representations under homomorphic images has been dealt with, in different contexts, in [25, 22] for CMILs and by Micol in [44] for $*$ -rings. Apparently, this needs saturation properties of ultrapowers. Considering a fixed 1-sorted algebraic structure A , add a new constant symbol \underline{a} , called a *parameter*, for each $a \in A$. In what follows, $\Sigma(x_1, \dots, x_n)$ is a set of formulas with free variables x_1, \dots, x_n in this extended language. Given an embedding $h: A \rightarrow B$, we call B

modestly saturated $[\omega\text{-saturated}]$ over A via h , if for any $n < \omega$ and for any set of formulas $\Sigma(x_1, \dots, x_n)$, with parameters from A [and finitely many parameters from B , respectively], which is finitely realized in A [in B , respectively] is realized in B (where \underline{a} is interpreted as $\underline{a}^B = h(a)$). The following is a particular case of [10, Corollary 4.3.14].

Proposition 8.4. *Every 1-sorted algebraic structure A admits an elementary embedding h into some structure B which is $[\omega\text{-}]$ saturated over A via h . One can choose B to be an ultrapower of A and h to be the canonical embedding. Identifying a with $h(a)$, one may assume B to be an elementary extension of A . The analogous result holds for multi-sorted algebraic structures.*

Theorem 8.5. *Let a CMIL L [$a * \Lambda$ -algebra R] have a faithful representation within a pre-hermitian space V_F . There is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L [such that for any regular ideal $I = I^*$, the algebra R/I] admits a faithful representation within $(U/\text{rad } U)_{\hat{F}}$, where $U = U^{\perp\perp}$ is a subspace of $\hat{V}_{\hat{F}}$.*

Proof. For a $*\Lambda$ -algebra R we use the same idea as in the proof of [30, Proposition 25]. Though here, the scalar product induced on U , as defined below, might be degenerated. According to Proposition 8.4, there is an ultrapower ${}_{\hat{R}}\hat{V}_{\hat{F}}$ of the faithful representation ${}_RV_F$ which is modestly saturated over ${}_RV_F$ via the canonical embedding. Then \hat{V} is an R -module via the canonical embedding of R into \hat{R} and the set

$$U = \{v \in \hat{V} \mid av = 0 \text{ for all } a \in I\} = \bigcap_{a \in I} (a^* \hat{V})^{\perp}$$

is a closed subspace of $\hat{V}_{\hat{F}}$ and a left (R/I) -module. Moreover as $I = I^*$, one has

$$\langle (r + I)v \mid w \rangle = \langle v \mid (r^* + I)w \rangle \text{ for all } v, w \in U \text{ and all } r \in R.$$

We observe that U^{\perp} is also an (R/I) -module. Indeed, if $v \in U^{\perp}$ then

$$\langle (r + I)v \mid u \rangle = \langle v \mid (r^* + I)u \rangle = 0 \text{ for all } u \in U.$$

Thus with $W = \text{rad } U$, one obtains an (R/I) - \hat{F} -bimodule U/W , where

$$(r + I)(v + W) = rv + W \text{ for all } r \in R \text{ and all } v \in U,$$

which is also a subquotient of V_F .

We show that ${}_{R/I}(U/W)_{\hat{F}}$ is a faithful representation of R/I ; that is, for any $a \in R \setminus I$, there has to be $u \in U$ such that $au \notin W$. It suffices to show that for any $a \in R \setminus I$, there are $u, v \in U$ such that $\langle au \mid v \rangle \neq 0$. Since $u \in U$ means $bu = 0$ for all $b \in I$, we have to show that the set

$$\Sigma(x, y) = \{\langle \underline{a}x \mid y \rangle \neq 0\} \cup \{\underline{b}x = 0 = \underline{b}y \mid b \in I\}$$

of formulas with parameters from $\{a\} \cup I$ and variables x, y of type V is satisfiable in ${}_{\hat{R}}\hat{V}_{\hat{F}}$. Due to saturation, it suffices to show that for any $b_1, \dots, b_n \in I$, there are $u, v \in V$ such that $\langle au \mid v \rangle \neq 0$ and $b_i u = b_i v = 0$ for all $i \in \{1, \dots, n\}$. In view of Proposition 3.1(iv) and regularity of I , there is an idempotent $e \in I$ such that $b_i e = b_i$ for all $i \in \{1, \dots, n\}$; in particular $b_i u = b_i v = 0$ whenever $eu = ev = 0$. Thus it suffices to show that there are $u, v \in V$ such that $eu = ev = 0$ but $\langle au \mid v \rangle \neq 0$.

Assume the contrary; namely, let $eu = ev = 0$ imply $\langle au \mid v \rangle = 0$ for all $u, v \in V$. For arbitrary $u', v' \in V$, let $u = (1 - e)u'$ and $v = (1 - e)v'$. As $eu = ev = 0$, we get by our assumption that $\langle (1 - e^*)au \mid v' \rangle = \langle au \mid v \rangle = 0$. This holds for all $v' \in V$,

whence $(1 - e^*)au = 0$ since V_F is non-degenerated. Thus $(1 - e^*)a(1 - e)u' = 0$ for all $u' \in V$, whence $(1 - e^*)a(1 - e) = 0$, as ${}_R V_F$ is a faithful representation. But then $a = e^*a + ae - e^*ae \in I$, a contradiction.

In the case of CMILs, given a representation $\varepsilon: L \rightarrow \mathbb{L}(V_F)$, let $G = \mathbb{G}(V_F)$ and let $\pi(v) = vF$ for $v \in V$. We consider the 4-sorted structure $(L, V, F, G; \varepsilon, \pi)$. According to Proposition 8.4, there is an ultrapower $(\hat{L}, \hat{V}, \hat{F}, \hat{G}; \hat{\varepsilon}, \hat{\pi})$ of $(L, V, F, G; \varepsilon, \pi)$ which is modestly saturated over $(L, V, F, G; \varepsilon)$ via the canonical embedding. By Lemma 8.3(i), $(\hat{L}, \hat{V}, \hat{F}; \hat{\varepsilon})$ is a faithful representation. In view of Proposition 7.2(ii), $\hat{G} \cong \mathbb{G}(\hat{V}_{\hat{F}})$ via $\hat{\pi}$; and $\hat{\rho}: W \mapsto \{v \in \hat{V} \mid \hat{\pi}(v) \in W\}$ defines an isomorphism from $\mathbb{L}(\hat{G})$ onto $\mathbb{L}(\hat{V}_{\hat{F}})$ by Proposition 7.2(iii).

Now, let θ be a congruence of the Galois lattice L . According to the proof of [22, Theorem 13.1], there is a faithful representation $\eta: L/\theta \rightarrow \mathbb{L}(W/W')$ in a subquotient W/W' of \hat{G} , where the subspace W is closed and $W' = W \cap W^\perp$. Then $\hat{\rho}(W)/\hat{\rho}(W')$ is a subquotient of $\hat{V}_{\hat{F}}$, $\hat{\rho}(W)$ is a closed subspace of \hat{V} , and $\hat{\rho}\eta$ is a faithful representation of L/θ in $\hat{\rho}(W)/\hat{\rho}(W')$ by Proposition 7.5. The proof is complete. \square

Corollary 8.6. *Let a MOL L have a faithful representation within a pre-hermitian space V_F . There is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L admits a faithful representation within a pre-hermitian closed subspace $U_{\hat{F}}$ of $\hat{V}_{\hat{F}}$.*

Proof. According to the proof of [22, Theorem 13.1] and the proof of Theorem 8.5, there is an ultrapower $\hat{V}_{\hat{F}}$ of V_F such that any homomorphic image of L admits a faithful representation within a subquotient W/W' of the orthogonal geometry $\mathbb{G}(\hat{V}_{\hat{F}})$. As L is a MOL, according to the definition of W' (given in [22, page 355] and denoted by U there), one has $W' = \emptyset$. Hence in the proof of Theorem 8.5, $\text{rad } U = \hat{\rho}(W') = 0$. \square

9. CLASSES OF STRUCTURES

We consider classes \mathcal{C} of $*$ - Λ -algebras on one side, Galois lattices on the other. With the familiar concepts, by $\mathbf{H}(\mathcal{C})$, $\mathbf{S}(\mathcal{C})$, $\mathbf{P}(\mathcal{C})$, $\mathbf{P}_s(\mathcal{C})$, $\mathbf{P}_\omega(\mathcal{C})$, and $\mathbf{P}_u(\mathcal{C})$, we denote the class of all homomorphic images, subalgebras, direct products, subdirect products, direct products of finitely many factors, and ultraproducts of members of \mathcal{C} , respectively, allowing isomorphic copies in all cases. Of course, all fundamental operations have to be taken care of. In particular, in the case of $*$ - Λ -algebras also the unit 1, the additive inverse, and the “scalars” $\lambda \in \Lambda$, that is, “subalgebra” means $*$ -subring and Λ -subalgebra. In the case of Galois lattices, also the bounds 0, 1 and the operation $x \mapsto x'$ are to be preserved, that is, “subalgebra” means Galois sublattice. In terms of Universal Algebra we have classes of algebraic structures of given “similarity type” or “signature” and the associated class operators, cf. [9, Chapter II] and [18, Chapter I], also [43].

A class \mathcal{C} of algebraic structures of the same type is a *universal class* if it is closed under \mathbf{S} and \mathbf{P}_u ; a *positive universal class* (shortly a *semivariety*), if it is closed also under \mathbf{H} ; a *variety* if, in addition, it is closed under \mathbf{P} . Let $\mathbf{W}(\mathcal{C})$ and $\mathbf{V}(\mathcal{C})$ denote the smallest semivariety and smallest variety containing \mathcal{C} . The following statement is well known and easily verified, cf. Theorem A.5 in Appendix A.

Proposition 9.1. *A class \mathcal{K} is universal [a semivariety, a variety] if and only if it can be defined by universal sentences [positive universal sentences, identities, respectively].*

Dealing with a class \mathcal{C} of $*$ - Λ -algebras or MILs, let $S_{\exists}(\mathcal{C})$ [$P_{s\exists}(\mathcal{C})$] consist of all regular or complemented members of the class $S(\mathcal{C})$ [of the class $P_s(\mathcal{C})$, respectively]. Call \mathcal{C} an \exists -*semivariety* if it is closed under the operators H , S_{\exists} , P_u and an \exists -*variety* if it is also closed under P , cf. [28], also [35] for an analogue within semigroup theory. Let $W_{\exists}(\mathcal{C})$ [$V_{\exists}(\mathcal{C})$] denote the least \exists -semivariety [\exists -variety, respectively] which contains the class \mathcal{C} .

Recall that MIL also denotes the class of all MILs, similarly for CMIL and MOL. Let \mathcal{A}_{Λ} denote the class of all $*$ - Λ -algebras, with the subclasses \mathcal{R}_{Λ} , \mathcal{R}_{Λ}^* , and \mathcal{F}_{Λ} consisting of its members which are regular, $*$ -regular, and divisions rings, respectively.

Proposition 9.2. *Let $\mathcal{C} \subseteq \mathcal{R}_{\Lambda}$ or $\mathcal{C} \subseteq \text{CMIL}$.*

- (i) $OS_{\exists}(\mathcal{C}) \subseteq S_{\exists}O(\mathcal{C})$ for any class operator $O \in \{P_u, P, P_{\omega}\}$.
- (ii) $S_{\exists}H(\mathcal{C}) \subseteq HS_{\exists}(\mathcal{C})$.
- (iii) $W_{\exists}(\mathcal{C}) = HS_{\exists}P_u(\mathcal{C})$.
- (iv) $V_{\exists}(\mathcal{C}) = HS_{\exists}P(\mathcal{C}) = HS_{\exists}P_uP_{\omega}(\mathcal{C}) = P_{s\exists}W_{\exists}(\mathcal{C})$.
- (v) $W_{\exists}(\mathcal{C})$ and $V_{\exists}(\mathcal{C})$ are axiomatic classes.
- (vi) $A \in W_{\exists}(\mathcal{C})$ if $B \in W_{\exists}(\mathcal{C})$ for all finitely generated $B \in S_{\exists}(A)$.

These statements are well known for arbitrary algebraic structures if the suffix \exists is omitted. For the proof of Proposition 9.2, we refer to the Appendix A.

Dealing with pre-hermitian spaces, we primarily adhere to the 2-sorted point of view as explained in Section 8. A (2-sorted) *embedding* $V'_{F'} \rightarrow V_F$ is given by a $*$ - Λ -algebra embedding $\alpha : F' \rightarrow F$ and an injective α -semilinear map ω such that $\langle \omega(v) \mid \omega(w) \rangle = \alpha(\langle v \mid w \rangle')$ for all $v, w \in V'$. An embedding is an *isomorphism* if both α and ω are bijections. $V'_{F'}$ is a (2-sorted) substructure of V_F if it embeds into V_F with α and ω being inclusion maps. In contrast, a *subspace* of V_F will always mean an F -linear subspace with the induced scalar product; that is, here we follow the 1-sorted view on the vector space V_F .

Let \mathcal{S} be a class of pre-hermitian spaces V_F , where $F \in \mathcal{F}_{\Lambda}$ and Λ is a fixed commutative $*$ -ring. In such a case, we also speak of a class of spaces *over* Λ . Introducing operators for classes of spaces, let $S(\mathcal{S})$ and $P_u(\mathcal{S})$ denote the classes of all non-degenerate 2-sorted substructures and all ultraproducts of members of \mathcal{S} respectively, or spaces which are isomorphic to such. In contrast to that, following the one-sorted view, let $S_{1f}(\mathcal{S})$ [$S_{1q}(\mathcal{S})$] denote the class of (isomorphic copies of) non-degenerate finite-dimensional subspaces [of all subquotients $U/\text{rad } U$ such that $V_F \in \mathcal{S}$, U_F is a subspace of V_F , and $U = U^{\perp\perp}$, respectively] of members of \mathcal{S} . The next statement follows from Propositions 2.2 and 2.3.

Lemma 9.3. *For any class \mathcal{S} of spaces over Λ , $S_{1f}(\mathcal{S}) \subseteq S_{1q}(\mathcal{S})$ and $S_{1f}S_{1q}(\mathcal{S}) = S_{1f}(\mathcal{S})$.*

Let also $I_s(\mathcal{S})$ denote the class of spaces which arise from \mathcal{S} by scaling and observe that $I_sO(\mathcal{S}) \subseteq OI_s(\mathcal{S})$ for any of the class operators introduced, above. Call \mathcal{S} a *universal class*, if it is closed under P_u , S , and I_s . Observe that $SP_uI_s(\mathcal{S})$ is the smallest universal class containing a class \mathcal{S} . Call \mathcal{S} a *semivariety* if it is closed under P_u and S_{1f} . Of course, any universal class is a semivariety, and the smallest semivariety containing a class \mathcal{S} is contained in $SP_u(\mathcal{S})$.

10. REDUCTION TO FINITE DIMENSION

Importance of representations for the universal algebraic theory of CMILs and regular *-rings derives from the following

Theorem 10.1. *Let V_F be a pre-hermitian space and let $L \in \text{MIL} [R \in \mathcal{A}_\Lambda]$ have a faithful representation within V_F . Then $L \in \text{W}(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)) [R \in \text{W}(\text{End}^*(U_F) \mid U \in \mathbb{O}(V_F))]$, respectively]. If $L \in \text{CMIL} [R \in \mathcal{R}_\Lambda]$, then $L \in \text{W}_\exists(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)) [R \in \text{W}_\exists(\text{End}^*(U_F) \mid U \in \mathbb{O}(V_F))]$, respectively].*

Proof. We may assume that $\dim V_F \geq \omega$. In view of Proposition 7.3, L embeds into an atomic subalgebra M of $\mathbb{L}(V_F)$ such that $M_f = \mathbb{L}(V_F)_f$ and M may be chosen a CMIL if L is such. Proposition 7.2(i)–(ii) yields that M_f is a CMIL and the directed union of its subalgebras $[0, U] \cup [U^\perp, V]$, $U \in \mathbb{O}(V_F)$, each of which is in $\text{S}_\exists(W_F)$ for some $W \in \mathbb{O}(V_F)$. Since any directed union of algebraic structures A_i embeds into an ultraproduct of the A_i (cf [18, Theorem 1.2.12(1)]), one gets

$$M_f \in \text{S}_\exists \text{P}_u(\mathbb{L}(U_F) \mid U \in \mathbb{O}(V_F)).$$

Finally, the proof of [22, Theorem 16.3] yields $M \in \text{W}(M_f)$ and $M \in \text{W}_\exists(M_f)$ if M is complemented. The claim about L follows, immediately.

Dealing with an algebra $R \in \mathcal{A}_\Lambda$, we first show that $\hat{J}(V_F) \in \text{W}_\exists(\text{End}^*(U_F) \mid U \in \mathbb{O}(V_F))$. By Proposition 9.2(vi), it suffices to prove this inclusion for finitely generated algebras $B \in \text{S}_\exists(\hat{J}(V_F))$. By Proposition 4.4(iv), we may assume that B is of the form $\{\varepsilon_U \varphi \pi_U + \lambda \text{id}_V \mid \varphi \in \text{End}^*(V_F), \lambda \in F\}$ for some $U \in \mathbb{O}(V_F)$. Thus $B \cong \text{End}^*(U_F) \times F$ and the latter embeds into $\text{End}^*(U_F) \times \text{End}^*((U^\perp \cap W)_F)$ which in turn into $\text{End}^*(W_F)$ for $U \subset W \in \mathbb{O}(V_F)$, cf. Propositions 2.2 and 4.3.

In view of Proposition 4.5, we may assume that R is a subalgebra of $\text{End}^*(V_F)$ containing $A = \hat{J}(V_F)$. Let $J = J(V_F)$ and let J_0 denote the set of projections in J . By Proposition 8.4, there is an ultrapower $(_{\hat{R}} \hat{V}_{\hat{F}}; \hat{A})$ of $(_R V_F; A)$ which is ω -saturated over $(_R V_F; A)$. We may assume that R is a subalgebra of \hat{R} and \hat{A} is an ultrapower of A ; in particular, $\hat{A} \in \text{W}_\exists(\text{End}^*(U_F) \mid U \in \mathbb{O}(V_F))$. For $a \in \hat{A}$ and $r \in R$, we put

$$a \sim r, \quad \text{if } ae = re \text{ and } a^*e = r^*e \text{ for all } e \in J_0.$$

Claim 1. *For any $a \in \hat{A}$ and any $r, s \in R$, $a \sim r$ and $a \sim s$ imply $r = s$.*

Proof of Claim. For any $U \in \mathbb{O}(V_F)$, we have $\pi_U \in J_0$, whence $r\pi_U = a\pi_U = s\pi_U$. Considering r and s as endomorphisms of V_F , we get that they coincide on any $U \in \mathbb{O}(V_F)$, whence they coincide on V_F by Proposition 2.2. \square

Claim 2. *$S = \{a \in \hat{A} \mid a \sim r \text{ for some } r \in R\}$ is a subalgebra of \hat{A} and the map*

$$g: \hat{A} \rightarrow R, \quad g: a \mapsto r, \text{ where } a \sim r$$

is a homomorphism.

Proof of Claim. It follows from Claim 1 that g is well-defined. Let $a, b \in \hat{A}$ and $r, s \in R$ be such that $a \sim r$ and $b \sim s$. Then, obviously, $a + b \sim r + s$, $\lambda a \sim \lambda r$ for any $\lambda \in \Lambda$, and $a^* \sim r^*$. Let $e \in J_0$, then $be \in J$. By Proposition 4.4(iv), there is $f \in J_0$ such that $fbe = be$. Therefore, we get $abe = afbe = rfbe = rbe = rse$, whence $ab \sim rs$.

Obviously, $0_{\hat{V}}, \text{id}_{\hat{V}} \in \hat{A}$. For any $U \in \mathbb{O}(V_F)$ we have $\pi_U \in J_0$. Therefore, $0_{\hat{V}}\pi_U = 0_U$ and $\text{id}_{\hat{V}}\pi_U = \pi_U$ imply in view of Proposition 2.2 that $0_{\hat{V}} \sim 0_R$ and $\text{id}_{\hat{V}} \sim 1_R$. \square

Claim 3. *The homomorphism g is surjective.*

Proof of Claim. Surjectivity of g is shown via the saturation property. Given $r \in R$, consider a finite set $E \subseteq J_0$. According to Proposition 4.4(iv), there is $e \in J_0$ such that $ef = f$ for all $f \in E$ and $er^*f = r^*f$ for all $f \in E$. Take $a = re$ and observe that $af = ref = rf$ and $a^*f = er^*f = r^*f$ for all $f \in E$. Thus the set of formulas

$$\Sigma(x) = \{[xe = re] \ \& \ [x^*e = r^*e] \mid e \in J_0\}$$

with a free variable x of type A is finitely realized in $({}_R V_F; A)$. As $({}_R \hat{V}_{\hat{F}}; \hat{A})$ is ω -saturated over $({}_R V_F; A)$, we get that there is $a \in \hat{A}$ with $a \sim r$. \square

Claim 4. *If R is regular, then S is also regular.*

Proof of Claim. In view of Proposition 3.1(ii), it suffices to prove that $\ker g = \{a \in S \mid a \sim 0\}$ is regular. Observe that $a \sim 0$ means that $ae = 0 = a^*e$ for any $e \in J_0$, equivalently $(1 - e)a = a = a(1 - e)$. Again, let $E \subseteq J_0$ be finite. By Proposition 4.4(iv), there is $e \in J_0$ such that $ef = f$ for any $f \in E$. The ring A is regular by Propositions 4.4(i) and 4.5, whence \hat{A} is also regular. Therefore, the ring $(1 - e)\hat{A}(1 - e)$ is regular by [5, 2.4]. Thus there is $b \in \hat{A}$ such that $aba = a$ and $(1 - e)b = b = b(1 - e)$; in particular, $be = 0 = eb$ whence $b^*e = 0$. This implies that $bf = bef = 0$ and $b^*f = b^*ef = 0$ for all $f \in E$. Therefore, the set of formulas

$$\Sigma(x) = \{axa = a\} \cup \{[xe = 0] \ \& \ [x^*e = 0] \mid e \in J_0\}$$

with a variable x of type A is finitely realized in $({}_R \hat{V}_{\hat{F}}; \hat{A})$. Thus $\Sigma(x)$ is realized in $({}_R \hat{V}_{\hat{F}}; \hat{A})$, and we obtain $b \in \hat{A}$ such that $aba = a$ and $b \sim 0$; that is, $b \in \ker g$. \square

The desired statements concerning $*$ - Λ -algebras follow from Claims 2-4 and Theorem A.5(iii). \square

Remark 10.2. The statements of Theorem 10.1 concerning $*$ - Λ -algebras were proved in case of representability in inner product spaces in [30, Theorem 16]. Requiring semivariety generation only, a more direct approach is possible. For $R \in \mathcal{A}_\Lambda$, one chooses in the proof of [30, Theorem 16] $I = \mathbb{O}(V_F)$. By Proposition 2.2, any finite-dimensional subspace of V_F is contained in some $U \in I$. Moreover, with the induced scalar product, U_F is a pre-hermitian space. A similar approach works for MILs.

11. (\exists) -SEMIVARIETIES OF REPRESENTABLE STRUCTURES

We denote by $\mathcal{L}(\mathcal{S})$ [$\mathcal{R}(\mathcal{S})$, respectively] the class of all CMILs [all $R \in \mathcal{R}_\Lambda$ respectively] having a faithful representation within some member of \mathcal{S} (we also say that these structures are *representable* within \mathcal{S}). We consider here conditions on \mathcal{S} which ensure that classes $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are \exists -(semi)varieties. Observe that

$$\mathcal{L}(\mathbf{l}_s(\mathcal{S})) = \mathcal{L}(\mathcal{S}) \quad \text{and} \quad \mathcal{R}(\mathbf{l}_s(\mathcal{S})) = \mathcal{R}(\mathcal{S}).$$

Proposition 11.1. *Let \mathcal{S} be a [recursively] axiomatized class of pre-hermitian spaces over a [recursive] commutative $*$ -ring Λ . Then $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are [recursively] axiomatizable.*

Proof. Let Γ_r denote the set of first order axioms defining representations ${}_R V_F$ with $R \in \mathcal{R}_\Lambda$ and $V_F \in \mathcal{S}$ and let Σ_r denote the set of all universal sentences in the signature of $*$ - Λ -algebras which are consequences of Γ_r . Then Σ_r defines the class of all $*$ - Λ -algebras representable in \mathcal{S} . Adding to Σ_r the $\forall\exists$ -axiom of regularity defines the subclass $\mathcal{R}(\mathcal{S})$. If Λ is recursive and \mathcal{S} is recursively axiomatizable, then Γ_r

is recursive. By Gödel's Completeness Theorem, Σ_r is recursively enumerable. By Craig's trick [32, Exercise 6.1.3], Σ_r is also recursive.

Similarly, taking Γ_l to be the set of first order axioms defining representations of CMILs within spaces from \mathcal{S} , and denoting by Σ_l the set of all universal sentences in the signature of CMILs which are consequences of Γ_l , we get that Σ_l defines the class $\mathcal{L}(\mathcal{S})$ of all CMILs representable in \mathcal{S} . Moreover, if Γ_l is recursive, then Σ_l is also recursive. We also refer to [42, 48]. \square

A *tensorial embedding* of a pre-hermitian space V_F into another one W_K is given by a $*$ - Λ -algebra embedding $\alpha: F \rightarrow K$ and an injective α -semilinear map $\varepsilon: V_F \rightarrow W_K$ such that W_K is spanned by $\text{im } \varepsilon$ as a K -vector space and $\langle \varepsilon(v) \mid \varepsilon(w) \rangle = \alpha(\langle v \mid w \rangle)$ for all $v, w \in V$; in particular, ε is an isomorphism of V_F onto a two-sorted substructure of W_K . A *joint tensorial extension* of spaces V_{iF_i} , $i \in \{0, 1\}$, is given by a pre-hermitian space $W_F = U_0 \oplus^\perp U_1$ and tensorial embedding of V_{iF_i} into U_{iF} for $i \in \{0, 1\}$.

Lemma 11.2. *Let $F, F_0, F_1 \in \mathcal{A}_\Lambda$, let V_F be a pre-hermitian space, and let V_{0F_0} and V_{1F_1} be finite-dimensional pre-hermitian spaces.*

- (i) *If α_i and ε_i define a tensorial embedding of V_{iF_i} into V_F , $i < 2$, then $\text{End}^*(V_{iF_i})$ embeds into $\text{End}^*(V_F)$ and $\mathbb{L}(V_{iF_i})$ embeds into $\mathbb{L}(V_F)$.*
- (ii) *If V_F is a joint tensorial extension of V_{0F_0} and V_{1F_1} , then $\text{End}^*(V_{0F_0}) \times \text{End}^*(V_{1F_1})$ embeds into $\text{End}^*(V_F)$ and $\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1})$ embeds into $\mathbb{L}(V_F)$.*

Proof. (i) In view of Proposition 4.3(i), V_{iF_i} has a dual pair $\{v_1, \dots, v_n\}, \{w_1, \dots, w_n\}$ of bases; applying ε_i , one obtains such a pair for V_F . Indeed, V_F is obviously spanned by both, $\{\varepsilon_i(v_1), \dots, \varepsilon_i(v_n)\}$ and $\{\varepsilon_i(w_1), \dots, \varepsilon_i(w_n)\}$. Suppose that $\sum_{j=1}^n \varepsilon_i(v_j) \lambda_j = 0$ for some $\lambda_1, \dots, \lambda_n \in F$. Then for any $k \in \{1, \dots, n\}$, one gets

$$0 = \langle 0 \mid w_k \rangle = \langle \sum_{j=1}^n \varepsilon_i(v_j) \lambda_j \mid \varepsilon_i(w_k) \rangle = \sum_{j=1}^n \alpha_i(\langle v_j \mid w_k \rangle) \lambda_j = \lambda_k,$$

whence $\{\varepsilon_i(v_1), \dots, \varepsilon_i(v_n)\}$ is a basis of V_F . Similarly, $\{\varepsilon_i(w_1), \dots, \varepsilon_i(w_n)\}$ is a basis of V_F .

For $\varphi \in \text{End}^*(V_{iF_i})$, let $\xi_i(\varphi)$ be the F -linear map on V defined by $\xi_i(\varphi): \varepsilon_i(v_j) \mapsto \varepsilon_i(\varphi(v_j))$ for all $j \in \{1, \dots, n\}$. Clearly, ξ_i is a Λ -algebra embedding of $\text{End}^*(V_{iF_i})$ into $\text{End}^*(V_F)$. Moreover, for any $j, k \in \{1, \dots, n\}$, one has

$$\begin{aligned} \langle \varepsilon_i(v_j) \mid \xi_i(\varphi^*) \varepsilon_i(v_k) \rangle &= \langle \varepsilon_i(v_j) \mid \varepsilon_i \varphi^*(v_k) \rangle = \alpha(\langle v_j \mid \varphi^*(v_k) \rangle) = \alpha(\langle \varphi(v_j) \mid v_k \rangle) = \\ &= \langle \varepsilon_i \varphi(v_j) \mid \varepsilon_i(v_k) \rangle = \langle \xi_i(\varphi) \varepsilon_i(v_j) \mid \varepsilon_i(v_k) \rangle, \end{aligned}$$

whence $\xi_i(\varphi^*) = \xi_i(\varphi)^*$ by Proposition 4.3(ii). For the claim about polarity lattices, apply Propositions 3.1(i), 5.4(iv), and 4.4(ii)-(iii).

- (ii) As $V_F = U_0 \oplus^\perp U_1$, by (i), there are $*$ - Λ -algebra embeddings

$$\xi_i: \text{End}^*(V_{iF_i}) \rightarrow \text{End}^*(U_{iF}), \quad i \in \{0, 1\}.$$

Thus there is a unique embedding

$$\xi: \text{End}^*(V_{0F_0}) \times \text{End}^*(V_{1F_1}) \rightarrow \text{End}^*(V_F)$$

such that $\xi(\varphi_0, \varphi_1)|_{U_i} = \xi_i(\varphi_i)$ for $i \in \{0, 1\}$. By Propositions 5.4(iii), 4.3(i), and 7.8(ii),

$$\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1}) \cong \mathbb{L}(\text{End}^*(V_{0F_0})) \times \mathbb{L}(\text{End}^*(V_{1F_1})) \cong \mathbb{L}(\text{End}^*(V_{0F_0}) \times \text{End}^*(V_{1F_1})).$$

By (i) and Proposition 7.8(i), the latter admits a faithful representation in V_F . \square

Theorem 11.3. *Let \mathcal{S} be a class of pre-hermitian spaces over Λ . Then*

- (i) $\mathcal{L}(\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S})) = \mathcal{L}(\mathcal{S}\mathcal{P}_u\mathcal{I}_s(\mathcal{S})) = \mathcal{W}_\exists(\mathcal{L}(\mathcal{S})) = \mathcal{W}_\exists(\mathbb{L}(V_F) \mid V_F \in \mathcal{S}_{1f}(\mathcal{S}));$
- (ii) $\mathcal{R}(\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S})) = \mathcal{R}(\mathcal{S}\mathcal{P}_u\mathcal{I}_s(\mathcal{S})) = \mathcal{W}_\exists(\mathcal{R}(\mathcal{S})) = \mathcal{W}_\exists(\text{End}^*(V_F) \mid V_F \in \mathcal{S}_{1f}(\mathcal{S})).$

In particular, if the class \mathcal{S} is a semivariety then the classes $\mathcal{L}(\mathcal{S}) = \mathcal{L}(\mathcal{S}\mathcal{P}_u\mathcal{I}_s(\mathcal{S}))$ and $\mathcal{R}(\mathcal{S}) = \mathcal{R}(\mathcal{S}\mathcal{P}_u\mathcal{I}_s(\mathcal{S}))$ are \exists -semivarieties generated by their strictly simple finite-dimensional or artinian members, respectively.

Proof. The proofs of (i) and (ii) follow the same lines. We prove (ii).

The fact that $\mathcal{S}_\exists\mathcal{P}_u(\mathcal{R}(\mathcal{S})) \subseteq \mathcal{R}(\mathcal{P}_u(\mathcal{S}))$ follows immediately from Lemma 8.3(iii). Then $\mathcal{W}_\exists(\mathcal{R}(\mathcal{S})) \subseteq \mathcal{R}(\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S}))$ by Theorem 8.5. By Theorem 10.1, $\mathcal{R}(\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S})) \subseteq \mathcal{W}_\exists(\text{End}^*(V_F) \mid V_F \in \mathcal{S}_{1f}\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S}))$. By Lemmas 8.3(iii) and 9.3, for any $V_F \in \mathcal{S}_{1f}\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S}) = \mathcal{S}_{1f}\mathcal{P}_u(\mathcal{S})$, we have $V_F \in \mathcal{P}_u\mathcal{S}_{1f}(\mathcal{S})$ and $\text{End}^*(V_F) \in \mathcal{P}_u(\text{End}^*(W_K) \mid W_K \in \mathcal{S}_{1f}(\mathcal{S}))$. It follows that

$$\mathcal{W}_\exists(\mathcal{R}(\mathcal{S})) \subseteq \mathcal{R}(\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S})) \subseteq \mathcal{W}_\exists(\text{End}^*(W_K) \mid W_K \in \mathcal{S}_{1f}(\mathcal{S})) \subseteq \mathcal{W}_\exists(\mathcal{R}(\mathcal{S})).$$

Now, consider $R \in \mathcal{R}(\mathcal{S}\mathcal{P}_u(\mathcal{S}))$; that is, R is represented in a 2-sorted substructure W_K of some $V_F \in \mathcal{P}_u(\mathcal{S})$. By Theorem 10.1, we have $R \in \mathcal{W}_\exists(\text{End}^*(U_K) \mid U_K \in \mathcal{S}_{1f}(W_K))$. Let U'_F denote the F -subspace of V_F spanned by U . By Lemma 11.2(i), $\text{End}^*(U_K) \in \mathcal{S}_\exists(\text{End}^*(U'_F))$. Thus, $R \in \mathcal{W}_\exists(\mathcal{R}(\mathcal{S}))$. Hence

$$\mathcal{R}(\mathcal{S}\mathcal{P}_u\mathcal{I}_s(\mathcal{S})) \subseteq \mathcal{R}(\mathcal{I}_s\mathcal{S}\mathcal{P}_u(\mathcal{S})) = \mathcal{R}(\mathcal{S}\mathcal{P}_u(\mathcal{S})) \subseteq \mathcal{W}_\exists(\mathcal{R}(\mathcal{S})) = \mathcal{R}(\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S})).$$

The containment $\mathcal{R}(\mathcal{S}_{1q}\mathcal{P}_u(\mathcal{S})) \subseteq \mathcal{R}(\mathcal{S}\mathcal{P}_u\mathcal{I}_s(\mathcal{S}))$ is trivial by Lemma 9.3. \square

More closure properties on \mathcal{S} are needed if one intends to get a one-to-one correspondence between classes of spaces and classes of structures in Theorem 11.3.

Definition 11.4. Let V_F, W_K be pre-hermitian spaces over Λ , $\dim V_F < \omega$, and let \mathcal{S} be a class of pre-hermitian spaces over Λ .

- (i) The sesquilinear space V_F is an L -spread of W_K if $\dim V_F > 2$ and $\mathbb{L}(V_F) \in \mathcal{L}(W_K)$. The class \mathcal{S} is L -spread closed, if it contains all L -spreads of its members.
- (ii) The sesquilinear space V_F is an R -spread of W_K if $\text{End}^*(V_F) \in \mathcal{R}(W_K)$. The class \mathcal{S} is R -spread closed, if it contains all R -spreads of its members.
- (iii) An R -[L]-spread closed universal class or a semivariety \mathcal{S} is *small*, if \mathcal{S} coincides with the smallest R -[L]-spread closed universal class or a semivariety which contains all members of \mathcal{S} of dimension $n < \omega$ [of dimension $2 < n < \omega$, respectively].

Example 11.5. Consider the class \mathcal{S} of all anisotropic hermitian spaces, where $F \in \mathcal{S}\mathcal{P}_u(\mathbb{Q})$; in particular, $F \models \forall x [x^2 \neq 2]$ and \mathcal{S} is a universal class which does not contain K_K^3 with the canonical scalar product, where $K = \mathbb{Q}(\sqrt{2})$. Nonetheless, $K^{3 \times 3}$ and whence $\mathbb{L}(K^{3 \times 3})$ is representable within $\mathbb{Q}_\mathbb{Q}^6 \in \mathcal{S}$ by

$$a + b\sqrt{2} \mapsto a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}; \quad \text{where } a, b \in \mathbb{Q},$$

which yields a $*$ -ring embedding of K into $\mathbb{Q}^{2 \times 2}$ thus giving rise to an embedding of $K^{3 \times 3}$ into $(\mathbb{Q}^{2 \times 2})^{3 \times 3}$. In the sense of Definition 11.4, K_K^3 is an L -spread and an R -spread of $\mathbb{Q}_\mathbb{Q}^6$.

Theorem 11.6.

- (i) For any \exists -semivariety \mathcal{V} of Arguesian CMILs generated by its strictly simple members of finite dimension at least 3, there is a small L -spread closed semivariety [universal class] \mathcal{S} of pre-hermitian spaces over \mathbb{Z} such that $\mathcal{V} = \mathcal{L}(\mathcal{S})$. Moreover, the class of members of \mathcal{S} of dimension at least 3 is unique.
- (ii) For any \exists -semivariety $\mathcal{V} \subseteq \mathcal{R}_\Lambda$ generated by its strictly simple artinian members, there is a small R -spread closed semivariety [universal class] \mathcal{S} of pre-hermitian spaces over Λ such that $\mathcal{V} = \mathcal{R}(\mathcal{S})$. Moreover, such a class \mathcal{S} is unique.

The class \mathcal{S} above is anisotropic, if \mathcal{V} consists of MOLs or $\mathcal{V} \subseteq \mathcal{R}_\Lambda^*$.

Remark 11.7. If \mathcal{V} consists of MOLs (context of (i)) or $\mathcal{V} \subseteq \mathcal{R}_\Lambda^*$ (context of (ii)) it suffices to require that \mathcal{V} is generated by its simple members which are of finite dimension respectively artinian and that, in context of (i), \mathcal{V} is not 2-distributive. Then, in context of (i), \mathcal{V} contains all MOLs of dimension 2.

Proof. (i) Given an \exists -semivariety $\mathcal{V} \subseteq \mathcal{R}_\Lambda$ with all required properties, let $\mathcal{K}_\mathcal{V}$ denote the class of strictly simple artinian members of \mathcal{V} . By Proposition 4.8, for any $R \in \mathcal{K}_\mathcal{V}$, there is a pre-hermitian space V_F over Λ such that $R \cong \text{End}^*(V_F)$. By $\mathcal{S}_\mathcal{V}$, we denote the class of spaces V_F over Λ such that $\text{End}^*(V_F) \in \mathcal{K}_\mathcal{V}$.

We put $\mathcal{G}_0 = \mathcal{S}_{\text{If}}(\mathcal{S}_\mathcal{V})$. For any ordinal α , let $\mathcal{G}_{\alpha+1}$ be the union of two classes: $\mathcal{P}_u(\mathcal{G}_\alpha)$ and the class of all $V_F \in \mathcal{S}_{\text{If}}(V'_F)$, where V'_F is an R -spread of some $W_K \in \mathcal{G}_\alpha$. Let also $\mathcal{G}_\alpha = \bigcup_{\beta < \alpha} \mathcal{G}_\beta$, if α is a limit ordinal.

Claim 1. $\mathcal{S}_{\text{If}}(\mathcal{G}_\alpha) \subseteq \mathcal{G}_\alpha$ and $\text{End}^*(V_F) \in \mathcal{V}$ for any α and $V_F \in \mathcal{G}_\alpha$ with $\dim V_F < \omega$.

Proof of Claim. We argue by induction on α . For $\alpha = 0$, the first claim follows from the definition of \mathcal{G}_0 . Moreover, if $U_F \in \mathcal{S}_{\text{If}}(V_F)$ and $\text{End}^*(V_F) \in \mathcal{V}$ then $\text{End}^*(U_F) \in \text{HS}_\exists(\text{End}^*(V_F)) \subseteq \mathcal{V}$ by Proposition 4.3(iii). The limit step is trivial. In the step from α to $\alpha + 1$, we assume first that V_F is isomorphic to an ultraproduct of spaces $V_{iF_i} \in \mathcal{G}_\alpha$, $i \in I$. If $U_F \in \mathcal{S}_{\text{If}}(V_F)$ and $n = \dim U_F$ then, by Lemma 8.3(iii), U_F is isomorphic to an ultraproduct of some $U_{iF_i} \in \mathcal{S}_{\text{If}}(V_{iF_i})$ with $\dim U_{iF_i} = n$, $i \in J$, for some $J \subseteq I$. By the inductive hypothesis, $U_{iF_i} \in \mathcal{G}_\alpha$ and $\text{End}^*(U_{iF_i}) \in \mathcal{V}$. Thus $U_F \in \mathcal{G}_{\alpha+1}$ and $\text{End}^*(U_F) \in \mathcal{V}$ by Lemma 8.3(iii).

Now, let V'_F be an R -spread of $W_K \in \mathcal{G}_\alpha$ and let $V_F \in \mathcal{S}_{\text{If}}(W_K)$. If $U_F \in \mathcal{S}_{\text{If}}(V_F)$ then $U_F \in \mathcal{S}_{\text{If}}(V'_F)$, whence $U_F \in \mathcal{G}_{\alpha+1}$ by definition. By Theorem 10.1 and the inductive hypothesis,

$$\text{End}^*(V'_F) \in \mathcal{W}_\exists(\text{End}^*(W'_K) \mid W'_K \in \mathcal{O}(W_K)) \subseteq \mathcal{V}.$$

By Proposition 7.2(i), $\text{End}^*(U_F) \in \text{HS}_\exists(\text{End}^*(V'_F)) \subseteq \mathcal{V}$. □

It follows that the R -spread closed semivariety $\mathbb{K}(\mathcal{V})$ of pre-hermitian spaces over Λ generated by $\mathcal{S}_\mathcal{V}$ is the union of the classes \mathcal{G}_α , where α ranges over all ordinals. Thus in view of the assumption $\mathcal{V} = \mathcal{W}_\exists(\mathcal{K}_\mathcal{V})$ and Claim 1, one gets by Theorem 11.3(ii)

$$\mathcal{V} \subseteq \mathcal{R}(\mathbb{K}(\mathcal{V})) = \mathcal{W}_\exists(\text{End}^*(V_F) \mid V_F \in \mathbb{K}(\mathcal{V}), \dim V_F < \omega) \subseteq \mathcal{V}.$$

To prove uniqueness, let \mathcal{S} and \mathcal{S}' be small R -spread closed semivarieties of pre-hermitian spaces over Λ such that $\mathcal{R}(\mathcal{S}) = \mathcal{V} = \mathcal{R}(\mathcal{S}')$. For any $V_F \in \mathcal{S}$ with $\dim V_F < \omega$, we have $\text{End}^*(V_F) \in \mathcal{R}(\mathcal{S}) = \mathcal{R}(\mathcal{S}')$, whence V_F is an R -spread of \mathcal{S}' and $V_F \in \mathcal{S}'$. Similarly, interchanging the roles of \mathcal{S} and \mathcal{S}' , we get that \mathcal{S} and \mathcal{S}' have the same artinian members.

To deal with the case of universal classes, one includes into the union \mathcal{G}_α a third class, namely $\mathbf{S}(\mathcal{G}_\alpha)$. Claim 1 and its proof remain valid, only the case of the third class remains to be considered. Indeed, assume that $V_F \in \mathcal{G}_{\alpha+1}$ is a 2-sorted substructure of $W_K \in \mathcal{G}_\alpha$ and let $U_F \in \mathbf{S}_{\text{lf}}(V_F)$. Then $U_F \in \mathbf{S}(W_K)$ and $U_F \in \mathcal{G}_{\alpha+1}$ by definition. Moreover, U_F is a 2-sorted substructure of the K -subspace U'_K of W_K spanned by U . In particular, $U'_K \in \mathbf{S}_{\text{lf}}(W_K)$ and the inductive hypothesis yields $U'_K \in \mathcal{G}_\alpha$ and $\mathbf{End}^*(U'_K) \in \mathcal{V}$. As $\mathbf{End}^*(U_F)$ embeds into $\mathbf{End}^*(U'_K)$ by Lemma 11.2(i), it follows that $\mathbf{End}^*(U_F) \in \mathcal{V}$.

(i) The proof follows the same lines as the one of (ii) replacing Λ by \mathbb{Z} , Proposition 4.8 by Proposition 7.7. Proposition 4.3(iii) by Proposition 7.2(i), and Theorem 11.3(ii) by Theorem 11.3(i). For $L \in \mathcal{K}_\mathcal{V}$ one has to require $3 \leq \dim L < \omega$. \square

For results of the same type as Theorem 11.6, see also [29, Theorems 4.4-5.4].

12. \exists -VARIETIES AND REPRESENTATIONS

We first consider a condition on \mathcal{S} under which the class of representables is an \exists -variety. Then we review the approach of Micol [44] to capture \exists -varieties via the concept of generalized representation.

A semivariety \mathcal{S} of pre-hermitian spaces over Λ is a *variety* if for any finite-dimensional $V_{0F_0}, V_{1F_1} \in \mathcal{S}$, there is a joint tensorial extension $V_F \in \mathcal{S}$.

Proposition 12.1. *If \mathcal{S} is a variety of pre-hermitian spaces over Λ , then $\mathcal{L}(\mathcal{S})$ and $\mathcal{R}(\mathcal{S})$ are \exists -varieties.*

Proof. In view of Proposition 9.2(iv) and Theorem 11.3, it suffices to notice that for any finite-dimensional spaces $V_{0F_0}, V_{1F_1} \in \mathcal{S}$, the structures $\mathbf{End}^*(V_{0F_0}) \times \mathbf{End}^*(V_{1F_1})$ and $\mathbb{L}(V_{0F_0}) \times \mathbb{L}(V_{1F_1})$ have a faithful representation within some member of \mathcal{S} by Lemma 11.2(ii). \square

Classes $\mathcal{L}(\mathcal{S})$ of CMILs having a faithful representation within some member of a class \mathcal{S} of orthogeometries have been considered in [22]. The closure properties of Theorem 11.3(i) hold also in this case with $\mathbf{S}(\mathcal{S})$ denoting formation of non-degenerate subgeometries of members of \mathcal{S} , $\mathbf{S}_{\text{lf}}(\mathcal{S})$ and $\mathbf{S}_{\text{la}}(\mathcal{S})$ — formation of non-degenerate finite-dimensional subspaces and of subquotients $U/\text{rad } U$, where $U_F \in \mathcal{S}$ and $U = U^{\perp\perp}$. In addition, one has the class $\mathbf{U}(\mathcal{S})$ of all disjoint orthogonal unions of members of \mathcal{S} and thus $\mathbf{P}(\mathcal{L}(\mathcal{S})) \subseteq \mathcal{L}(\mathbf{U}(\mathcal{S}))$, cf. [22, Theorem 2.2]. Moreover, mimicking the concept of an L -spread and the proof of Theorem 11.6, one obtains

Theorem 12.2. *For any \exists -variety \mathcal{V} of CMILs generated by its finite-dimensional members, there is a small L -spread and \mathbf{U} -closed semivariety [universal class] \mathcal{S} of orthogeometries such that $\mathcal{V} = \mathcal{L}(\mathcal{S})$. Moreover, such a class \mathcal{S} is unique.*

The objective of Micol [44] was to derive results for $*$ -regular rings, analogous to those above. Of course, representation requires some structure of the type of sesquilinear spaces. Apparently, in general there is no axiomatic class of such spaces which would serve for representing direct products of representable structures. Micol solved this problem by introducing the concept of a *generalized representation*. This concept was transferred to MOLs by Niemann [46].

A *g-representation* of $A \in \text{CMIL}$ [$A \in \mathcal{R}_\Lambda$] within a class \mathcal{S} of pre-hermitian spaces is a family $\{\varepsilon_i \mid i \in I\}$ of representations ε_i of A in $V_{iF_i} \in \mathcal{S}$, $i \in I$. It is *faithful* if $\bigcap_{i \in I} \ker \varepsilon_i$ is trivial. Let $\mathcal{L}_g(\mathcal{S})$ [$\mathcal{R}_g(\mathcal{S})$] denote the class of all $A \in \text{CMIL}$ [$A \in \mathcal{R}_\Lambda$] having a faithful g -representation within \mathcal{S} ; equivalently, the class of structures A

having a subdirect decomposition into factors $\varepsilon_i(A)$, $i \in I$, which have a faithful representation within \mathcal{S} .

Call an artinian algebra $R \in \mathcal{R}_\Lambda$ *strictly artinian* if $I = I^*$ for any ideal I of R . By the Wedderburn-Artin Theorem, this is equivalent to the fact that R is isomorphic to a direct product of strictly simple factors (cf. [38, §3.4]). Similarly, call a finite-dimensional CMIL L *strictly finite-dimensional* if $\theta = \theta'$ for any lattice congruence θ of L . By [6, Theorem IV.7.10]), this is equivalent to the fact that L is a direct product of strictly simple factors.

Proposition 12.3. *The following statements are true.*

- (i) *For any semivariety \mathcal{S} of pre-hermitian spaces, the class $\mathcal{L}_g(\mathcal{S}) = \text{P}_{\exists\exists}(\mathcal{L}(\mathcal{S}))$ [$\mathcal{R}_g(\mathcal{S}) = \text{P}_{\exists\exists}(\mathcal{R}(\mathcal{S}))$] is an \exists -variety generated by its strictly simple finite-dimensional [artinian] members, which are of the form $\mathbb{L}(V_F)$ [$\text{End}^*(V_F)$] with $V_F \in \mathcal{S}$, $\dim V_F < \omega$.*
- (ii) *For any \exists -variety $\mathcal{V} \subseteq \text{CMIL}$ [$\mathcal{V} \subseteq \mathcal{R}_\Lambda$] which is generated by its strictly finite-dimensional at least 3 [artinian] members, there is a semivariety \mathcal{S} of pre-hermitian spaces such that $\mathcal{V} = \mathcal{L}_g(\mathcal{S})$ [$\mathcal{V} = \mathcal{R}_g(\mathcal{S})$].*
- (iii) *$A \in \mathcal{L}_g(\mathcal{S})$ [$A \in \mathcal{R}_g(\mathcal{S})$] if and only if A has an atomic extension \hat{A} which is a subdirect product of atomic strictly subdirectly irreducible structures A_i such that $(A_i)_f \cong \mathbb{L}(V_{iF_i})_f$ [the minimal ideal of A_i is isomorphic to $J(V_{iF_i})$] with $V_{iF_i} \in \mathcal{S}$.*

Proof. Statement (i) follows from Propositions 9.2(iii)-(iv), 7.7, 4.8, and Theorem 11.3. Statement (ii) follows from Propositions 9.2(iv), 7.7, 4.8, and Theorem 11.6. Finally, statement (iii) follows from Propositions 7.3, 4.5 and Theorems 7.6, 4.6. \square

For *-regular rings, the result of Proposition 12.3 is in essence due to Micol [44]. To prove that g -representability is preserved under homomorphic images, she axiomatized families of inner product spaces as 3-sorted structures, where the third sort mimics the index set I . Again, a saturation property is needed for the proof and regularity is crucial. The fact that the \exists -variety of g -representable structures is generated by its artinian members was shown by her reducing to countable subdirectly irreducible structures R , deriving countably based representation spaces (and forming 2-sorted subspaces), and using the approach of Tyukavkin [50] with respect to a countable orthogonal basis. Conversely, a substantial part of Theorem 11.3 follows from Proposition 12.3.

APPENDIX A. EXISTENCE SEMIVARIETIES

We characterize \exists -(semi)varieties contained in CMIL or in \mathcal{R}_Λ as model classes, proving at the same time the operator identities of Proposition 9.2. With no additional effort, this can be done to include other classes of algebraic structures.

Given a set Σ of first order axioms, by $\text{Mod } \Sigma$ we denote the model class $\{A \mid A \models \Sigma\}$ of Σ . By $\text{Th } \mathcal{C}$ [$\text{Th}_L \mathcal{C}$], we denote the set of sentences [from the fragment L] of first order language which are valid in \mathcal{C} . As usual, let \bar{x} denote a sequence of variables of length being given by context.

Definition A.1. A class \mathcal{C}_0 of algebraic structures of the same similarity type is *regular* if there is a (possibly empty) set Ψ_0 of conjunctions $\alpha(\bar{x}, y)$ of atomic formulas (i.e. formulas of the form $\bigwedge_{i=1}^k s_i(\bar{x}, y) = t_i(\bar{x}, y)$) and a class \mathcal{S} such that

- (i) $\mathcal{C}_0 = \mathcal{S} \cap \text{Mod}\{\forall \bar{x} \exists y \alpha(\bar{x}, y) \mid \alpha(\bar{x}, y) \in \Psi_0\}$;

- (ii) \mathcal{S} is closed under \mathbf{S} and \mathcal{C}_0 is closed under \mathbf{H} and \mathbf{P} ;
- (iii) For any structures $A, B \in \mathcal{C}_0$, for any surjective homomorphism $\varphi: A \rightarrow B$, for any formula $\alpha(\bar{x}, y) \in \Psi_0$, and for any $\bar{a}, b \in B$ such that $B \models \alpha(\bar{a}, b)$, there are $\bar{c}, d \in A$ such that $\varphi(\bar{c}) = \bar{a}$, $\varphi(d) = b$, and $A \models \alpha(\bar{c}, d)$.

Without loss of generality, one may consider also the case when $\alpha(\bar{x}, y)$ is of the form $\bigwedge_{i=1}^k p_i(t_1(\bar{x}, y), \dots, t_m(\bar{x}, y))$, where p_i is a relation symbol of arity m or the symbol $=$ with $m = 2$.

From Definition A.1(ii) it follows immediately that any regular class is closed under \mathbf{P}_u . In the sequel, we shall fix a regular class \mathcal{C}_0 and write for any $\mathcal{C} \subseteq \mathcal{C}_0$:

$$\mathbf{S}_{\exists}(\mathcal{C}) = \mathcal{C}_0 \cap \mathbf{S}(\mathcal{C}) \quad \text{and} \quad \mathbf{P}_{\mathbf{S}\exists}(\mathcal{C}) = \mathcal{C}_0 \cap \mathbf{P}_{\mathbf{S}}(\mathcal{C}).$$

Let \mathcal{C}_0 be a regular class. A *Skolem expansion* A^* of $A \in \mathcal{C}_0$ adds for each $\alpha(\bar{x}, y) \in \Psi_0$ an operation f_α on A such that $A \models \alpha(\bar{a}, f_\alpha(\bar{a}))$ for all $\bar{a} \in A$.

Definition A.2. A class \mathcal{C}_0 is *strongly regular* if it is regular and

- (iii') For any structures $A, B \in \mathcal{C}_0$, for any surjective homomorphism $\varphi: A \rightarrow B$, for any formula $\alpha(\bar{x}, y) \in \Psi_0$, for any $\bar{a}, b \in B$ such that $B \models \alpha(\bar{a}, b)$, and for any $\bar{c} \in A$ such that $\varphi(\bar{c}) = \bar{a}$ there is $d \in A$ such that $\varphi(d) = b$ and $A \models \alpha(\bar{c}, d)$.

Remark A.3. It is obvious that if a class \mathcal{C}_0 satisfies (iii') of Definition A.2, then \mathcal{C}_0 satisfies (iii) of Definition A.1. For any strongly regular class \mathcal{C}_0 , for any $A, B \in \mathcal{C}_0$, and for any surjective homomorphism $\varphi: A \rightarrow B$, if B^* is a Skolem expansion of B , then there is a Skolem expansion A^* of A such that $\varphi: A^* \rightarrow B^*$ is a homomorphism. Clearly, \mathcal{C}_0 is strongly regular if it satisfies (i)-(ii) of Definition A.1 and for any $\alpha \in \Psi_0$ and for any $\bar{a} \in A \in \mathcal{C}_0$, there is unique b such that $\alpha(\bar{a}, b)$. This applies, in particular, to completely regular [inverse] semigroups.

In what follows, when we speak of a [strongly] regular class \mathcal{C} , we always assume that the set of formulas Ψ_0 and the classes \mathcal{C}_0 and \mathcal{S} are given according to Definition A.1 [Definition A.2, respectively].

Proposition A.4. *For any variety \mathcal{V} with a $*$ -ring reduct, the class of structures $A \in \mathcal{V}$ having $*$ -regular reducts forms a strongly regular class. In particular, the class \mathcal{R}_Λ^* of all $*$ -regular $*$ - Λ -algebras is strongly regular.*

Proof. Let $\Psi_0 = \{xyx = y\}$ and let $\mathcal{S} = \mathcal{V} \cap \text{Mod}(\forall x \, xx^* = 0 \rightarrow x = 0)$. Then \mathcal{C}_0 defined as in Definition A.1(i) consists of the $*$ -regular members of \mathcal{V} . Closure of \mathcal{C}_0 under \mathbf{H} and \mathbf{P} follows from the fact that $*$ -regularity can be defined by the sentence:

$$\forall x \exists y (y = y^2 = y^*) \ \& \ (\exists u \, x = uy) \ \& \ (\exists u \, y = ux).$$

The proof of (iii') essentially goes as in [17, Lemma 1.4], cf. [28, Lemma 9]. Indeed, the two-sided ideal $I = \ker \varphi$ is regular. Let $c \in A$ be such that $a = \varphi(c)$, and let $aba = a$ in B . There is $y \in A$ such that $\varphi(y) = b$. Then $c - cyc \in I$. Since I is regular, there is $u \in I$ such that $(c - cyc)u(c - cyc) = c - cyc$. It follows from the latter that $cuc - cycuc - cucyc + cycucyc + cyc = c$. Taking $d = u - ucy - ycu + ycucy + y$, we get $cdc = cuc - cucyc - cycuc + cycucyc + cyc = c$ and $d - y = u - ucy - ycu + ycucy \in I$, whence $\varphi(d) = b$. \square

Further examples of strongly regular classes are the class of all regular [complemented] members of any variety having ring [bounded modular lattice, respectively]

reducts, see [28, Lemma 9]. The latter can be easily modified to the class of all relatively complemented lattices; here $\alpha(x_1, x_2, x_3, y)$ is given by $y((x_1 + x_2)x_3 + x_1x_2) = x_1x_2 \ \& \ y + ((x_1 + x_2)x_3 + x_1x_2) = x_1 + x_2$.

We consider fragments of the first order language associated with a given regular class \mathcal{C}_0 . Let L_u consist of all quantifier free formulas; up to equivalence, we may assume that L_u consists of conjunctions of formulas $\bigwedge_{i=1}^n \beta_i \rightarrow \bigvee_{j=1}^m \gamma_j$, where β_i, γ_j are atomic formulas and $n, m \geq 0$. The set $L_q \subseteq L_u$ of all *quasi-identities* is defined by $m = 1$. The set L_p consists of all formulas of the form

$$\bigwedge_{i=1}^n \alpha_i(\bar{x}_i, y_i) \rightarrow \bigvee_{j=1}^m \gamma_j,$$

where $n \geq 0, m \geq 1$, and $\alpha_i(\bar{x}_i, y_i) \in \Psi_0$. Then $L_e \subseteq L_p$ is defined by $m = 1$; its members are called *conditional identities*, while those of L_p are *conditional disjunctions of equations*. As usual, validity of a formula means validity of its universal closure. We write Th_x instead of Th_{L_x} .

Theorem A.5. *Let \mathcal{C}_0 be a regular class and let $\mathcal{C} \subseteq \mathcal{C}_0$. Then the following equalities and definabilities are granted.*

- (i) $\mathcal{C}_0 \cap \text{Mod Th}_u \mathcal{C} = S_{\exists} P_u(\mathcal{C})$. *In particular, \mathcal{C} is definable by universal sentences relatively to \mathcal{C}_0 if and only if it is closed under S_{\exists} and P_u .*
- (ii) $\mathcal{C}_0 \cap \text{Mod Th}_q \mathcal{C} = S_{\exists} P_u P_{\omega}(\mathcal{C}) = S_{\exists} P P_u(\mathcal{C})$. *In particular, \mathcal{C} is definable by quasi-identities relatively to \mathcal{C}_0 if and only if it is closed under S_{\exists} , P_u , and P_{ω} [under S_{\exists} , P_u , and P , respectively].*
- (iii) $\mathcal{C}_0 \cap \text{Mod Th}_p \mathcal{C} = H S_{\exists} P_u(\mathcal{C})$. *In particular, \mathcal{C} is definable by conditional disjunctions of equations relatively to \mathcal{C}_0 if and only if it is closed under H , S_{\exists} , and P_u .*
- (iv) $\mathcal{C}_0 \cap \text{Mod Th}_e \mathcal{C} = H S_{\exists} P_u P_{\omega}(\mathcal{C}) = H S_{\exists} P P_u(\mathcal{C})$. *In particular, \mathcal{C} is definable by conditional identities relatively to \mathcal{C}_0 if and only if it is closed under H , S_{\exists} , P_u , and P_{ω} [under H , S_{\exists} , P , and P_u , respectively].*

Classes as in (iii) and (iv) will be called \exists -semivarieties and \exists -varieties, respectively. If Ψ_0 is empty, one has *semivarieties* and *varieties*. By $W_{\exists}(\mathcal{C})$ [by $V_{\exists}(\mathcal{C})$, $W(\mathcal{C})$, $V(\mathcal{C})$, respectively], we denote the smallest \exists -semivariety [\exists -variety, semivariety, variety, respectively] containing \mathcal{C} , cf. Theorem A.5(iii)-(iv).

Of course, the statements of Theorem A.5 are well known results in the case of empty Ψ_0 . Proofs of (i) and (ii) are included since they can be seen as a preparation for proofs of (iii)-(iv); the latter are our primary interest.

Proof. Inclusion in the model class is well known and easy to verify in any of the cases (i)-(iv) using Definition A.1. In particular in cases (iii)-(iv), inclusion $H(\mathcal{C}) \subseteq \text{Mod Th}_x \mathcal{C}$ follows directly from Definition A.1(iii).

The proof of the reverse inclusion relies on adapting the method of diagrams. Given a structure A , let $a \mapsto x_a$ be a bijection onto a set of variables and let $\bar{x} = (x_a \mid a \in A)$. We consider quantifier free formulas $\chi(\bar{x})$ in these variables; evaluations \bar{x} in a structure B are given as $\bar{b} = (b_a \mid a \in A) \in B^A$, and we write $B \models \chi(\bar{b})$ if $\chi(\bar{x})$ is valid under evaluation \bar{b} . For a set $\Phi = \Phi(\bar{x})$ of formulas, $B \models \Phi(\bar{b})$ if $B \models \chi(\bar{b})$ for all $\chi(\bar{x}) \in \Phi$. Let At denote the set of atomic formulas

and let

$$\begin{aligned}\Delta^+(A) &= \{\chi(\bar{x}) \in At \mid A \models \chi(\bar{a})\}; \\ \Delta^-(A) &= \{\neg\chi(\bar{x}) \mid \chi(\bar{x}) \in At, A \not\models \chi(\bar{a})\}; \\ \Delta^0(A) &= \left\{ \alpha(t_1(\bar{x}), \dots, t_n(\bar{x}), x_a) \mid \right. \\ &\quad \left. t_1, \dots, t_n \text{ are terms, } \alpha(x_1, \dots, x_n, y) \in \Psi_0, A \models \alpha(t_1(\bar{a}), \dots, t_n(\bar{a}), a) \right\};\end{aligned}$$

$$\Delta_u(A) = \Delta_q(A) = \Delta^+(A) \cup \Delta^-(A);$$

$$\Delta_p(A) = \Delta_e(A) = \Delta^0(A) \cup \Delta^-(A).$$

For $x \in \{u, q, p, e\}$ and a finite subset Φ of $\Delta_x(A)$, let $\Phi^- = \Phi \cap \Delta^-(A)$, $\Phi^+ = \Phi \setminus \Phi^-$, and let Φ^\dagger denote the formula

$$\bigwedge_{\varphi \in \Phi^+} \varphi \rightarrow \bigvee_{\neg\chi \in \Phi^-} \chi;$$

while for $\neg\chi \in \Phi^-$, let Φ_χ^\dagger denote the quasi-identity

$$\bigwedge_{\varphi \in \Phi^+} \varphi \rightarrow \chi.$$

Thus for any finite $\Phi \subseteq \Delta_u(A)$ and for $\chi \in \Phi^-$, we have $\Phi^\dagger \in L_u$ and $\Phi_\chi^\dagger \in L_q$, while for any finite $\Phi \subseteq \Delta_p(A)$ and for $\chi \in \Phi^-$, we have $\Phi^\dagger \in L_p$ and $\Phi_\chi^\dagger \in L_e$. Observe that $A \not\models \Phi^\dagger$ and $A \not\models \Phi_\chi^\dagger$ in any case (verified by substituting x_a with a). Let $A \in \mathcal{C}_0 \cap \text{Mod Th}_x \mathcal{C}$. We have to obtain A from \mathcal{C} by means of operators.

First, we consider the case $x \in \{u, p\}$. Let $\Phi \subseteq \Delta_x(A)$ be finite. As $A \not\models \Phi^\dagger$, we have that $\Phi^\dagger \notin \text{Th}_x \mathcal{C}$. Thus there are a structure $B_\Phi \in \mathcal{C}$ and $\bar{b}_\Phi = (b_{\Phi a} \mid a \in A) \in B_\Phi^A$ such that $B_\Phi \not\models \Phi^\dagger(\bar{b}_\Phi)$, i.e. $B_\Phi \models \Phi(\bar{b}_\Phi)$.

As in the proof of the Compactness Theorem, let I be the set of all finite subsets of $\Delta_x(A)$ and let \mathcal{U} be one of the ultrafilters containing all sets $\{\Psi \in I \mid \Psi \supseteq \Phi\}$, where $\Phi \in I$. Let $B = \prod_{\Phi \in I} B_\Phi / \mathcal{U}$, $b_a = (b_{\Phi a} \mid \Phi \in I) / \mathcal{U}$ and $\bar{b} = (b_a \mid a \in A)$. By (the quantifier free part of) the Łoś Theorem, we have $B \models \Delta_x(A)(\bar{b})$. Moreover, $B \in \mathbf{P}_u(\mathcal{C}) \subseteq \mathcal{C}_0$.

Let C be the subalgebra of B generated by the set $\{b_a \mid a \in A\}$. We claim that $C \in \mathcal{C}_0$, i.e. $C \in \mathbf{S}_\exists(B)$. Indeed, let $\alpha(x_1, \dots, x_n, y) \in \Psi_0$ and let $c_1, \dots, c_n \in C$. As C is generated by the set $\{b_a \mid a \in A\}$, there are terms $t_1(\bar{x}), \dots, t_n(\bar{x})$ such that $c_i = t_i(\bar{b})$ for all $i \in \{1, \dots, n\}$. Since $A \in \mathcal{C}_0$, by Definition A.1(i) there is $a \in A$ such that

$$A \models \alpha(t_1(\bar{a}), \dots, t_n(\bar{a}), a).$$

Therefore,

$$\alpha(t_1(\bar{x}), \dots, t_n(\bar{x}), x_a) \in \Delta^+(A) \cap \Delta^0(A).$$

Since $B \models \Delta_x(A)(\bar{b})$, we conclude that $B \models \alpha(t_1(\bar{b}), \dots, t_n(\bar{b}), b_a)$. This implies that $C \models \alpha(c_1, \dots, c_n, b_a)$. On the other hand, $B \in \mathbf{P}_u(\mathcal{C}) \subseteq \mathcal{C}_0 \subseteq \mathcal{S}$, as \mathcal{C}_0 is closed under \mathbf{P}_u by Definition A.1(ii). Therefore, $C \in \mathbf{S}(B) \subseteq \mathbf{S}(\mathcal{S}) \subseteq \mathcal{S}$ again by Definition A.1(ii). This implies by Definition A.1(i) that $C \in \mathcal{C}_0$ which is our desired conclusion. Furthermore, the map

$$\varphi: C \rightarrow A; \quad t(\bar{b}) \mapsto t(\bar{a})$$

is well-defined (since $B \models \Delta^-(A)(\bar{b})$), a homomorphism (in view of term composition), and surjective (since $\varphi(b_a) = a$). Moreover, in case $x = u$, φ is an isomorphism, as $B \models \Delta^+(A)(\bar{b})$. This proves (i) and (iii).

Let $x \in \{q, e\}$. Given a finite subset $\Phi \subseteq \Delta_x(A)$ and $\neg\chi \in \Phi^-$, one has $A \not\models \Phi_\chi^\dagger$, whence $\Phi_\chi^\dagger \notin \text{Th}_x \mathcal{C}$. Thus there are a structure $B_{\Phi, \chi} \in \mathcal{C}$ and $\bar{b}_{\Phi\chi} = (b_{\Phi\chi a} \mid a \in A) \in B_{\Phi, \chi}^A$ such that

$$B_{\Phi, \chi} \models \Phi^+(\bar{b}_{\Phi\chi}) \quad \text{and} \quad B_{\Phi, \chi} \models \neg\chi(\bar{b}_{\Phi\chi}).$$

Taking $B_\Phi = \prod_{\neg\chi \in \Phi^-} B_{\Phi, \chi} \in \mathbf{P}_\omega(\mathcal{C})$ and $b_{\Phi a} = (b_{\Phi\chi a} \mid \neg\chi \in \Phi^-)$, we get that $B_\Phi \models \Phi(\bar{b}_\Phi)$. As above, let $B = \prod_{\Phi \in I} B_\Phi / \mathcal{U}$, $b_a = (b_{\Phi a} \mid \Phi \in I) / \mathcal{U}$, so that $B \models \Delta_x(A)(\bar{b})$. Let C be again the subalgebra of B generated by the set $\{b_a \mid a \in A\}$. We get as above that $C \in \mathbf{S}_\exists \mathbf{P}_u \mathbf{P}_\omega(\mathcal{C})$. Thus $A \in \mathbf{H}(C)$ for $x = e$ and $A \cong C$ for $x = q$ follow exactly as above.

It remains to show that $A \in \mathbf{HS}_\exists \mathbf{PP}_u(\mathcal{C})$ if $x = e$ and $A \in \mathbf{S}_\exists \mathbf{PP}_u(\mathcal{C})$ if $x = q$. Here, we fix $\neg\chi \in \Delta(A)^-$ and consider the set $I_\chi = \{\Phi \in I \mid \neg\chi \in \Phi^-\}$. Then there is a non-principal ultrafilter \mathcal{U}_χ on I which contains all sets $\{\Psi \in I_\chi \mid \Psi \supseteq \Phi\}$ with $\Phi \in I_\chi$. Take

$$B_\chi = \prod_{\Phi \in I_\chi} B_{\Phi, \chi} / \mathcal{U}_\chi; \quad b_{\chi a} = (b_{\Phi\chi a} \mid \Phi \in I_\chi) / \mathcal{U}_\chi; \quad \bar{b}_\chi = (b_{\chi a} \mid a \in A),$$

so that $B_\chi \models \neg\chi(\bar{b}_\chi)$ and $B_\chi \models \Delta^+(A)(\bar{b}_\chi)$ if $x = q$, $B_\chi \models \Delta^0(A)(\bar{b}_\chi)$ if $x = e$. Then

$$B' = \prod_{\neg\chi \in \Delta^-(A)} B_\chi \in \mathbf{PP}_u(\mathcal{C}); \quad B' \models \Delta_x(A)(\bar{b}'), \quad \text{where } b'_a = (b_{\chi a} \mid \chi \in \Delta^-(A)).$$

Let C' be the subalgebra of B' generated by the set $\{b'_a \mid a \in A\}$. As above, $C' \in \mathbf{S}_\exists(B')$ and $A \in \mathbf{H}(C')$ (if $x = e$) or $A \cong C'$ (if $x = q$) via the map $\varphi'(t(\bar{b}')) = t(\bar{a})$. The proof is now complete. \square

The following recaptures [28, Proposition 10]. For convenience, we include proofs.

Proposition A.6. *Let \mathcal{C}_0 be a strongly regular class and let $\mathcal{C} \subseteq \mathcal{C}_0$.*

- (i) $\mathbf{S}_\exists \mathbf{H}(\mathcal{C}) \subseteq \mathbf{HS}_\exists(\mathcal{C})$;
- (ii) $\mathbf{V}_\exists(\mathcal{C}) = \mathbf{HS}_\exists \mathbf{P}(\mathcal{C})$;
- (iii) *If all members of \mathcal{C}_0 have a distributive congruence lattice, then $A \in \mathbf{W}_\exists(\mathcal{C})$ for any subdirectly irreducible structure $A \in \mathbf{V}_\exists(\mathcal{C})$.*

Proof. (i) Let structures A , B and C be such that $A \in \mathcal{C}$, $C \in \mathbf{S}_\exists(B)$, and let $\varphi: A \rightarrow B$ be a surjective homomorphism. Then $B, C \in \mathcal{C}_0$ by Definition A.1(ii). Choose a Skolem expansion C^* of C and extend it to a Skolem expansion B^* of B . According to Remark A.3, there is a Skolem expansion A^* of A such that $\varphi: A^* \rightarrow B^*$ is a homomorphism. Then $C^* \in \mathbf{S}(B^*) \subseteq \mathbf{SH}(A^*) \subseteq \mathbf{HS}(A^*)$, whence $C^* \in \mathbf{H}(D^*)$ for some $D^* \in \mathbf{S}(A^*)$ and $C \in \mathbf{H}(D)$ with $D \in \mathbf{S}_\exists(A)$.

(ii) According to Theorem A.5(iv), $\mathbf{V}_\exists(\mathcal{C}) = \mathbf{HS}_\exists \mathbf{PP}_u(\mathcal{C})$. Straightforward inclusions $\mathbf{P}_u(\mathcal{C}) \subseteq \mathbf{HP}(\mathcal{C})$ and $\mathbf{PH}(\mathcal{C}) \subseteq \mathbf{HP}(\mathcal{C})$ together with (i) imply:

$$\mathbf{V}_\exists(\mathcal{C}) \subseteq \mathbf{HS}_\exists \mathbf{PHP}(\mathcal{C}) \subseteq \mathbf{HS}_\exists \mathbf{HP}(\mathcal{C}) \subseteq \mathbf{HS}_\exists \mathbf{P}(\mathcal{C}).$$

The reverse inclusion is obvious.

(iii) Let $A \in \mathbf{V}_\exists(\mathcal{C})$ be subdirectly irreducible. Then by (ii), there is $B \in \mathbf{S}_\exists \mathbf{P}(\mathcal{C})$ such that $A \in \mathbf{H}(B)$. By Jónsson's Lemma, there is $C \in \mathbf{SP}_u(\mathcal{C})$ such that $A \in \mathbf{H}(C)$ and $C \in \mathbf{H}(B)$. The latter inclusion implies by Definition A.1(ii) that $C \in \mathcal{C}_0$, whence $C \in \mathbf{S}_\exists \mathbf{P}_u(\mathcal{C})$. \square

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